Vortex motion

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July 1, 2016

1 The concept of vorticity

We call

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}$$

vorticity. It is a measure of the "swirlyness" of the flow, but is also present in shear flows where the velocity vector points uniformly in one direction, but its magnitude varies in a direction orthogonal to it.

In the following an incompressible flow with constant density and therefore $\nabla \cdot \mathbf{v} = 0$ will always be assumed if not stated differently.

Given $\boldsymbol{\omega}$, the corresponding velocity field is given by the Biot-Savart law

$$\mathbf{v}(\mathbf{x}) = \frac{1}{4\pi} \int \mathrm{d}^3 x' \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} \tag{1}$$

Note that this formula implies $\mathbf{x}' \in \mathbb{R}^3$ imperatively.

Having the evolution of the velocity governed by the incompressible Navier-Stokes equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{\rho} = \eta \Delta \mathbf{v} \tag{2}$$

one obtains an evolution equation for the vorticity by applying the curl operator to it:

$$\partial_t \boldsymbol{\omega} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = \eta \Delta \boldsymbol{\omega} \tag{3}$$

$$\partial_t \boldsymbol{\omega} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} + \eta \Delta \boldsymbol{\omega} \tag{4}$$

Here the identities

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \mathbf{v} \times \boldsymbol{\omega} \tag{5}$$

$$\nabla \times (\mathbf{v} \times \omega) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - \boldsymbol{\omega}(\nabla \cdot v) + v(\nabla \cdot \omega) \tag{6}$$

and $\nabla \times \Delta v = \Delta(\nabla \times v)$ have been used. Using the material derivative $\mathrm{d}_t := \partial_t + \mathbf{v} \cdot \nabla$ one thus has

$$\boxed{\mathrm{d}_t \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} + \eta \Delta \boldsymbol{\omega}} \tag{7}$$

This equation deserves a number of comments:

- Observe that in absence of viscosity ($\eta = 0$), if the vorticity vanishes everywhere in a fluid at rest at $t = 0$, then it will remain zero for all later times even if the velocity will not remain zero.

- The rather particular source term $(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}$ vanishes if the flow is purely two-dimensional. Indeed, if $\mathbf{v}$ depends on x and y only and does not have a z-component[1], then $\boldsymbol{\omega} = (\partial_x v_y - \partial_y v_x)\mathbf{e}_z$.

- Despite the additional source term one can show, in absence of viscosity, the advection of vorticity with the fluid in a certain sense. Consider the *circulation*

$$\Gamma = \oint \mathbf{v}\cdot \mathrm{d}\mathbf{l} \tag{8}$$

with the integral extending over a closed loop. If the closed loop is taken to move alongside with the fluid, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma = \oint (\mathrm{d}_t\mathbf{v})\cdot\delta\mathbf{l} + \oint \mathbf{v}\cdot\delta(\mathrm{d}_t\mathbf{l}) \tag{9}$$

$$= -\frac{1}{\rho}\oint \nabla p\cdot\delta\mathbf{l} + \oint \mathbf{v}\cdot\delta\mathbf{v} = 0 \tag{10}$$

where now the infinitesimal quantity involved in integration has been given the letter δ, in order to distinguish it from the differentiation with time. Both integrals vanish upon integration over a closed loop, which proves *Kelvin's theorem*. The integral can be rewritten as

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial S}\mathbf{v}\cdot\mathrm{d}\mathbf{l} = \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{\omega}\cdot\mathrm{d}\mathbf{s} \tag{11}$$

It expresses the conservation of the flux of vorticity through any surface whose boundaries move alongside with the flow. We see here that vortex filaments cannot have dangling ends, but must either close upon itself[2] or end on boundaries.

- In presence of a nonvanishing source term $(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}$ its effects remind of the behaviour of a gyroscope or other rotating bodies. In particular it is tightly related to the conservation of angular momentum. The source term can be split into two contributions: one parallel to the local direction of ω and one perpendicular to it. Consider at a given point in space the z-axis to be parallel to $\boldsymbol{\omega}$ and therefore v_z to be the component of the velocity which is parallel to it. Then $\mathbf{v}_\perp := \mathbf{v} - \mathbf{e}_z v_z$ will be perpendicular to $\boldsymbol{\omega}$. In total:

$$(\boldsymbol{\omega}\cdot\nabla)\mathbf{v} = \omega\partial_z\mathbf{v} = \omega\partial_z v_z\mathbf{e}_z + \omega\partial_z\mathbf{v}_\perp \tag{12}$$

The first term expresses conservation of angular momentum. $\partial_z v_z \neq 0$ means that the vortex filament is being stretched, and as the mass inside it must remain constant this implies a decrease in cross-section. Conservation of angular momentum thus enforces an increase in vorticity, a result also directly visible form Kelvin's theorem in Eq. (11).

[1] In all of this document, without any exception, an index is never meant to indicate a derivative.

[2] Indeed it was possible to create even knotted vortices: doi:10.1038/nphys2560.

2 2-dimensional vortex flows

Note that the Biot-Savart formula becomes

$$\mathbf{v}(\mathbf{r}) = \frac{1}{2\pi} \int \mathrm{d}^2 r' \frac{\boldsymbol{\omega}(\mathbf{r}') \times (\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^2} \tag{13}$$

if the variable z is integrated out and now $\mathbf{r}' \in \mathbb{R}^2$.

2.1 Cylindrically symmetric vorticity distribution

In cylindrical coordinates, the curl is given by

$$\boldsymbol{\omega} = \left(\frac{1}{r} \partial_\varphi v_z - \partial_z v_\varphi \right) \mathbf{e}_r + (\partial_z v_r - \partial_r v_z)\, \mathbf{e}_\varphi + \frac{1}{r} \left(\partial_r (r v_\varphi) - \partial_\varphi v_r \right) \mathbf{e}_z \tag{14}$$

If the velocity is a function of r only, and points solely in φ-direction, then

$$\boldsymbol{\omega} = \frac{1}{r} \partial_r (r v_\varphi) \mathbf{e}_z \tag{15}$$

In general, if $\mathbf{v} = v_0 r^n \mathbf{e}_\varphi$, then $\boldsymbol{\omega} = v_0 (n+1) r^{n-1}$. No vorticity implies $n = -1$, whereas constant vorticity needs $n = 1$.

Consider a Gaussian distribution of vorticity

$$\boldsymbol{\omega} = \omega_0 \exp\left(-\frac{r^2}{r_0^2} \right) \mathbf{e}_z \tag{16}$$

Then one finds the corresponding velocity to be

$$v_\varphi = \frac{1}{2} \omega_0 r_0 \frac{1 - \exp\left(-\frac{r^2}{r_0^2} \right)}{r/r_0} \simeq \begin{cases} \frac{1}{2} \omega_0 r & r \ll r_0 \\ \frac{1}{2} \omega_0 r_0 \frac{r_0}{r} & r \gg r_0 \end{cases} \tag{17}$$

where the boundary condition of vanishing velocity at the origin has been used. One observes a velocity $\propto \frac{1}{r}$ in the nearly irrotational region, and a solid-body rotation in the vicinity of the origin.

2.2 Point vortices

Letting $r_0 \to 0$, such that $\omega_0 r_0^2$ stays finite, results in a *point vortex* in 2-d or a *straight line vortex* in 3-d. There the vorticity is localized in the origin, and vanishes everywhere else in the fluid. As then, apart from a finite number of such singularities, the bulk of the fluid moves in an irrotational manner, line vortices can be treated nicely in an analytic manner. However, as the space is not simply-connected any more, a potential for the velocity exists only locally.

Therefore the circulation around such a vortex will not vanish:

$$\Gamma = \int_0^{2\pi} \frac{v_0}{r} r \mathrm{d}\varphi = 2\pi v_0 \tag{18}$$

It is customary to make this circulation explicit by writing the velocity of a single vortex of *strength* Γ as

$$\mathbf{v} = \frac{\Gamma}{2\pi r}\mathbf{e}_\varphi \tag{19}$$

The vortex moves with the local velocity of the fluid, but it vanishes at its location, as could be seen from its construction as the limit of finite size vorticity distribution. Therefore such a vortex remains stationary, and the fluid moves in circles around the singularity.

Consider now two such vortices at locations $\mathbf{x}_A$ and $\mathbf{x}_B$ and of strengths Γ_A and Γ_B, respectively. The motion of vortex A is entirely given by the velocity which B induces at $\mathbf{x}_B$. Therefore it is always perpendicular to $\mathbf{x}_A - \mathbf{x}_B$. Calling their distance $d := |\mathbf{x}_A - \mathbf{x}_B|$, one easily finds that if $\Gamma_A = -\Gamma_B =: \Gamma$, the whole system moves in a straight line with speed $\Gamma/(2\pi d)$, and if $\Gamma_A = \Gamma_B =: \Gamma$, then the two vortices orbit each other with a speed $\Gamma/(2\pi d)$. In general,

$$\dot{\mathbf{x}}_A = \frac{\Gamma_B}{2\pi d^2}\mathfrak{R}(\mathbf{x}_B - \mathbf{x}_A) \qquad \dot{\mathbf{x}}_B = -\frac{\Gamma_A}{2\pi d^2}\mathfrak{R}(\mathbf{x}_B - \mathbf{x}_A) \tag{20}$$

where $\mathbb{R} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ is the rotation by 90 in positive direction. Then we seek to find a frame moving with constant velocity $\frac{U}{d}\mathfrak{R}(\mathbf{x}_B - \mathbf{x}_A)$, such that in this frame the velocities of A and B are equal in absolute value and opposite in direction. This means

$$\frac{\Gamma_B}{2\pi d^2} - \frac{U}{d} = -\left(-\frac{\Gamma_A}{2\pi d^2} - \frac{U}{d}\right) \tag{21}$$

$$U = \frac{\Gamma_B - \Gamma_A}{4\pi d} \tag{22}$$

The orbital speed of the two vortices in this frame is

$$\frac{\Gamma_A + \Gamma_B}{4\pi d} \tag{23}$$

Exercise The behaviour of a vortex of strength Γ located at $\mathbf{x}$ in the vicinity of two walls forming a right angle is the subject of study in this exercise. Take the walls to be given by the positive parts of the x and y axes; there the normal velocity has to vanish.

i) Argue that the behaviour of this vortex can be studied by considering a suitably chosen system of 4 point vortices in an unbounded domain. Make a sketch of the arrangement and the sense of rotation of the fluid around the four vortices.

ii) Show the velocity $\dot{\mathbf{x}}$ to be proportional to

$$\frac{1}{r^2}\begin{pmatrix} x^2/y \\ -y^2/x \end{pmatrix} \tag{24}$$

where $r^2 = x^2 + y^2$.

iii) Show that $\mathbf{x} \times \dot{\mathbf{x}}$ is constant. (This is called *angular momentum conservation.*)

iv) This allows to conclude that the motion due to a central force, i.e.

$$\ddot{\mathbf{x}} = f(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \tag{25}$$

Compute f.

Solution

i) The vortices have to be arranged by mirroring the one given, and the rotation arrangement is

$$\begin{array}{c|c} -\Gamma & \Gamma \\ \hline \Gamma & -\Gamma \end{array} \tag{26}$$

ii) We have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{\Gamma}{4\pi x}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{\Gamma}{4\pi y}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\Gamma}{4\pi(x^2+y^2)}\begin{pmatrix} -y \\ x \end{pmatrix} \tag{27}$$

$$= \frac{\Gamma}{4\pi r^2}\begin{pmatrix} x^2/y \\ -y^2/x \end{pmatrix} \tag{28}$$

iii)

$$\mathbf{x} \times \dot{\mathbf{x}} = \mathbf{e}_z(x\dot{y} - y\dot{x}) = -\mathbf{e}_z\frac{\Gamma}{4\pi} \tag{29}$$

iv)

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{\Gamma}{4\pi}\begin{pmatrix} \frac{2x\dot{x}y^3 - x^2\dot{y}x^2 - x^2y^2\dot{y} - 2x^2y^2\dot{y}}{y^2(x^2+y^2)^2} \\ \cdots \end{pmatrix} = \frac{3\Gamma^2}{(4\pi)^2 r^3}\frac{\mathbf{x}}{r} \tag{30}$$

and thus $f(r) = \frac{3\Gamma^2}{(4\pi)^2}\frac{1}{r^3}$. Note that the force is repulsive irrespective of the sign of Γ. As you might recall from your course on classical mechanics, such a force does not produce close orbits (Betrand's theorem). However you might also have encountered it as the centrifugal potential in a Kepler problem, and therefore it is not too surprising that it comes across here again.

2.3 Rows of vortices

It is possible to compute the motion of infinite rows of vortices, which can be thought of as approximation to vortex streets seen in the wake of objects moving through a fluid. This is treated in the book by Lamb. Many arrangements turn out to be unstable, the only stable one being that of a double row with certain spacings between the vortices and a particular alignment of the circulations. However these stability results can altered by taking into account additional spatial dimensions and an internal structure of the vortex (instead of it being a singularity). Also due to its complexity despite all the simplifying assumptions this analysis is outside of the scope of these lecture notes.

2.4 Non-singular vorticity distributions

In this section only a particular result can be treated. Suppose the fluid to have a uniform vorticity ω in the region $\mathcal{E} = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$ with $a > b$ and to be irrotational outside it. It turns out that the motion of this setup is a steady rotation of the ellipse without change of shape.

Recall that the stream function ψ allows to encode divergencelessness of the flow: upon differentiation of

$$v_x := -\partial_y \psi \qquad\qquad v_y := \partial_x \psi \tag{31}$$

one obtains $\partial_x v_x + \partial_y v_y \equiv 0$. Inside $\mathcal{E}$ we have

$$\omega = \partial_x v_y - \partial_y v_x = (\partial_x^2 + \partial_y^2)\psi \tag{32}$$

Making the ansatz $\psi = Ax^2 + By^2$, one has

$$\omega = 2(A + B) \tag{33}$$

We don't have uniformly $v_x = -\Omega y$ and $v_y = \Omega x$, but it is only the ellipse which rotates in this manner. Any tangential velocity would not be visible in such a rotation therefore this assumption cannot yield complete information on the velocity field. It is only known that

$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \begin{pmatrix} -\Omega y \\ \Omega x \end{pmatrix} \tag{34}$$

where $\mathbf{n}$ is the normal onto the ellipse.

Given a function $f(x,y)$ its gradient points orthogonally to the lines of constant f, and thus is proportional to $\mathbf{n}$:

$$\frac{x v_x}{a^2} + \frac{y v_y}{b^2} = \Omega x y \left(-\frac{1}{a^2} + \frac{1}{b^2} \right) \tag{35}$$

Inserting the ansatz yields

$$2Aa^2 - 2Bb^2 = \Omega \left(a^2 - b^2 \right) \tag{36}$$

Continuity of the tangential velocity at the boundary of $\mathcal{E}$ requires the knowledge of the irrotational flow outside. This is a topic by itself, which therefore here will be treated only briefly. The usage of the stream function turns out to be of great value again. Note first that the derivative of x and y with respect to the arc length s evaluated along a curve yields the tangential, and upon rotation, also the normal vectors:

$$\begin{pmatrix} t_x \\ t_y \end{pmatrix} = \begin{pmatrix} \partial_s x \\ \partial_s y \end{pmatrix} = \begin{pmatrix} n_y \\ -n_x \end{pmatrix} \tag{37}$$

Therefore, just as we have evaluated it above, the velocity in the direction of the normal equals to the velocity of the boundary in that direction. The above calculation was needed in just a special case, here is the general case for rotations:

$$\begin{pmatrix} -\partial_y\psi \\ \partial_x\psi \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} -\Omega y \\ \Omega x \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_y \end{pmatrix} \tag{38}$$

$$\partial_s\psi = \partial_s y\partial_y\psi + \partial_s x\partial_x\psi = \Omega(y\partial_s y + x\partial_s x) = \frac{1}{2}\Omega\partial_s(x^2+y^2) \tag{39}$$

$$\psi = \frac{1}{2}\Omega(x^2+y^2) + \text{const} \tag{40}$$

where the last equations has been obtained by integrating along the boundary. Given any ψ the above equation will yield a family of curves which, upon rotation with angular velocity Ω would create a flow field given by that ψ. We now just have to be lucky to find a ψ which would yield an ellipse. Lamb (§71,3 and §72,2) starts from such a ψ and shows it to be indeed the motion of an irrotational fluid due to a rotating ellipse. Here we will confine ourselves to stating the result:

$$\psi = \frac{1}{4}\Omega(a+b)^2e^{-2\xi}\cos(2\eta) + \frac{1}{2}\omega ab\xi \tag{41}$$

with

$$x = \sqrt{a^2-b^2}\cosh\xi\cos\eta \tag{42}$$

$$y = \sqrt{a^2-b^2}\sinh\xi\sin\eta \tag{43}$$

The latter are so-called *elliptic coordinates*. For constant ξ this evidently is a parametrization of an ellipse. You can show these coordinates to be orthogonal, such that the normal onto any such fixed ellipse is proportional to

$$\begin{pmatrix} \partial_\xi x \\ \partial_\xi y \end{pmatrix} \tag{44}$$

and the tangential velocity equals

$$\begin{pmatrix} -\partial_\xi y \\ \partial_\xi x \end{pmatrix} \cdot \begin{pmatrix} -\partial_y\psi \\ \partial_x\psi \end{pmatrix} = \partial_\xi y\partial_y\psi + \partial_\xi x\partial_x\psi = \partial_\xi\psi \tag{45}$$

Equating now these tangential components of the velocity given by the irrotational flow outside and the rotational flow inside the ellipse one obtains

$$-\frac{1}{2}\Omega(a+b)^2e^{-2\xi}\cos(2\eta) + \frac{1}{2}\omega ab = \left(\frac{\partial x}{\partial\xi}, \frac{\partial y}{\partial\xi}\right)\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}(Ax^2+By^2) \tag{46}$$

$$= 2\sqrt{a^2-b^2}(\sinh\xi\cos\eta Ax + \cosh\xi\sin\eta By) \tag{47}$$

$$= 2(a^2-b^2)(A\cos^2\eta + B\sin^2\eta) \tag{48}$$

$$= (a^2-b^2)\sinh\xi\cosh\xi\Big(A(\cos(2\eta)+1) + B(1-\cos(2\eta))\Big) \tag{49}$$

We only need this at the actual location of the ellipse. Given a and b, the value of ξ satisfies

$$\cosh\xi = \frac{a}{\sqrt{a^2-b^2}} \qquad \sinh\xi = \frac{b}{\sqrt{a^2-b^2}} \tag{50}$$

and thus, using $e^{-2\xi} = \frac{a-b}{a+b}$

$$-\frac{1}{2}\Omega(a^2 - b^2) = ab(A - B) \tag{51}$$

Equations (33), (36) and (51) are sufficient to determine A, B and Ω. In particular:

$$A = \omega\frac{b}{2(a+b)} \qquad B = \frac{a\omega}{2(a+b)} \qquad \Omega = \omega\frac{ab}{(a+b)^2} \tag{52}$$

3 3-dimensional vortex flows

3.1 Self-interaction of curved vortex filaments

Consider a vortex filament of constant circulation Γ and width ξ which is curved with a radius of curvature $R \gg \xi$. Take one point of this filament to be in the origin of the coordinate system. The velocity induced by the vortex filament onto it is given by

$$\mathbf{v} = \frac{\Gamma}{4\pi}\int \frac{\mathrm{d}\mathbf{x} \times \mathbf{x}}{|\mathbf{x}|^3} \tag{53}$$

Placing the filament into the x-y-plane, one has

$$\mathbf{x} = R\begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} \qquad \mathrm{d}\mathbf{x} = R\mathrm{d}\varphi\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \tag{54}$$

It turns out that the integral becomes singular around $\varphi = 0$. However this also means that the highest contribution to the velocity at a point of the vortex comes from neighbouring points. Therefore, an idea of this self-interaction can be obtained by linearizing around $\varphi = 0$, but removing the interval $[-\xi/R, \xi/R]$ from the integration domain, which up to numeric constants of the order 1 gives

$$|\mathbf{v}| \simeq \frac{\Gamma}{\pi R}\ln\left(\frac{R}{\xi}\right) \tag{55}$$

3.2 Circular vortex

The above estimate has shown that for curved vortex filaments the approximation of a singular vorticity does not lead to interesting results as then the macroscopic properties of the motion of the vortices also diverge. This makes analysis of these structures quite complicated. An interesting example is that a vortex ring, which is a vortex filament of size ξ and radius R that can easily be produced by smokers in the form of a smoke ring. Due to decreased pressure inside the vortex the smoke stays inside and travels alongside with the motion of the fluid. Just as a pair of point vortices, a smoke ring cannot be stationary. Its velocity at every of its points is given by the influence of the others. A rough estimate for this speed was given above. A circular cross-section with radius $\xi/2$ yields the more precise estimate

$$v = \frac{\Gamma}{4\pi R}\ln\left(\frac{16R}{\xi} - \frac{1}{4}\right) \tag{56}$$

for its speed of propagation (Lamb §163, Eq. (7)). The analysis is also made difficult by the fact that the ring preserves its shape during the motion only approximately. The question of motion of a vortex ring without change of shape leads to approximately elliptic cross-sections for vortex rings with small ξ. A particularly tractable example of such a vortex is given in the section below.

An interesting effect concerns a pair of coaxial vortices with parallel velocities. The influence of the front vortex is to shrink the vortex behind it and to pass it through itself, as the speed of the vortex ring is larger when the size is smaller. The vortex behind is enlarging the vortex in front of it and slowing it down. Once the vortex from behind has appeared in the front, the process repeats. This leap-frog effect can be readily observed in experiments.

3.3 Hill's spherical vortex

Exercise Consider a flow with cylindrical symmetry, i.e. the components v_r and v_z depend only on $r = \sqrt{x^2 + y^2}$ and z, and $v_\varphi = 0$. The spherical radius will be denoted by $r_\mathrm{s} := \sqrt{x^2 + y^2 + z^2}$.

i) Show that $\boldsymbol{\omega}$ has only a component along $\mathbf{e}_\varphi$. This component is denoted by ω in the following.

ii) Starting from $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$ show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega}{r}\right)\mathbf{e}_\varphi = 0 \tag{57}$$

The slightly different form of the divergence operator needs a modified definition of the stream function ψ:

iii) Find $A(r)$ and $B(r)$ such that

$$v_r = A(r)\partial_z\psi \qquad\qquad v_z = B(r)\partial_r\psi \tag{58}$$

implies $\nabla \cdot \mathbf{v} = 0$. Take $A(r)$ to be positive.

iv) Show that now

$$\omega = \frac{1}{r}\left(\partial_z^2\psi + \partial_r^2\psi - \frac{1}{r}\partial_r\psi\right) \tag{59}$$

v) Take $\psi = \frac{1}{2}Ar^2(R^2 - r_\mathrm{s}^2)$ inside a sphere of radius R with A, R constants (this is the stream function of Hill's spherical vortex). Show that $\frac{\omega}{r}$ is a uniform constant and therefore fulfills the equations.

The irrotational flow of a stream with speed $-U\mathbf{e}_z$ past a sphere of radius R is given by

$$\psi = \frac{1}{2}Ur^2\left(1 - \frac{R^3}{r_\mathrm{s}^3}\right) \tag{60}$$

vi) Show that for $r_\mathrm{s} \to \infty$ the velocity approaches $-U\mathbf{e}_z$.

vii) By patching together the two solutions and enforcing continuity of the tangential velocity (i.e. continuity of $\frac{\partial\psi}{\partial r_s}$) express ω as a function of U and r. Use $r = r_s \sin\vartheta$, where ϑ is the usual angle encountered in spherical polar coordinates.

By changing the frame of reference this gives a "spherical" vortex ring advancing at speed U in a fluid which is at rest at infinity.

Hints:

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\partial_\varphi A_z - \partial_z A_\varphi\right)\mathbf{e}_r + \left(\partial_z A_r - \partial_r A_z\right)\mathbf{e}_\varphi + \frac{1}{r}\left(\partial_r\left(rA_\varphi\right) - \partial_\varphi A_\rho\right)\mathbf{e}_z \tag{61}$$

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = \left(A_r\partial_r B_r + \frac{A_\varphi}{r}\partial_\varphi B_r + A_z\partial_z B_r - \frac{A_\varphi B_\varphi}{r}\right)\mathbf{e}_r \tag{62}$$

$$+\left(A_r\partial_r B_\varphi + \frac{A_\varphi}{r}\partial_\varphi B_\varphi + A_z\partial_z B_\varphi + \frac{A_\varphi B_r}{r}\right)\mathbf{e}_\varphi \tag{63}$$

$$+\left(A_r\partial_r B_z + \frac{A_\varphi}{r}\partial_\varphi B_z + A_z\partial_z B_z\right)\mathbf{e}_z \tag{64}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\partial_r\left(rA_r\right) + \frac{1}{r}\partial_\varphi A_\varphi + \partial_z A_z \tag{65}$$

Solution

i) Clearly $\boldsymbol{\omega} = \left(\partial_z v_r - \partial_r v_z\right)\mathbf{e}_\varphi$.

ii)

$$\partial_t\boldsymbol{\omega} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} = 0 \tag{66}$$

$$\partial_t\omega\mathbf{e}_\varphi + \left(v_r\partial_r\omega + v_z\partial_z\omega\right)\mathbf{e}_\varphi - \left(\frac{\omega v_r}{r}\right)\mathbf{e}_\varphi = 0 \tag{67}$$

$$\partial_t\frac{\omega}{r^2} + v_r\frac{\partial_r\omega}{r} + v_z\partial_z\frac{\omega}{r} - \frac{\omega v_r}{r^2} = 0 \tag{68}$$

$$\Big(\partial_t + (\mathbf{v}\cdot\nabla)\Big)\frac{\omega}{r}\mathbf{e}_\varphi = 0 \tag{69}$$

iii) As

$$\frac{1}{r}\partial_r\left(rA(r)\partial_z\psi\right) + B(r)\partial_z\partial_r\psi = 0 \tag{70}$$

one should take $A(r) = \frac{1}{r}$, $B(r) = -\frac{1}{r}$.

iv) clear.

v) Upon direct computation one finds

$$\omega = -5Ar \tag{71}$$

Note: Lamb (1932 edition) seems to have a typo here (between formulae (14) and (15) on p. 245, §165).

vi)

$$\psi = \frac{1}{2}Ur^2\left(1 - \frac{R^3}{(r^2+z^2)^{3/2}}\right) \tag{72}$$

$$v_r = \frac{1}{r}\partial_z\psi = \frac{3}{2}Ur\frac{zR^3}{(r^2+z^2)^{5/2}} \overset{r_{\rm s}\to\infty}{\longrightarrow} 0 \tag{73}$$

$$v_z = -\frac{1}{r}\partial_r\psi = -U\left(1 - \frac{R^3}{(r^2+z^2)^{3/2}}\right) - \frac{3}{2}U\frac{r^2R^3}{(r^2+z^2)^{5/2}} \overset{r_{\rm s}\to\infty}{\longrightarrow} -U \tag{74}$$

If $r_{\rm s} = R$:

$$v_r = \frac{3}{2}U\frac{r}{R^2}z \qquad\qquad v_z = -\frac{3}{2}U\frac{r}{R^2}r \tag{75}$$

$$\mathbf{v}\cdot\begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix} = \frac{3}{2}U\frac{R\sin\vartheta}{R^2}R\cos\vartheta\sin\vartheta - \frac{3}{2}U\frac{R^2\sin^2\vartheta}{R^2}\cos\vartheta = 0 \tag{76}$$

Therefore indeed at the boundary of the sphere the velocity is wholly tangential to it.

vii)

$$\psi = \frac{1}{2}Ur_{\rm s}^2\sin^2\vartheta\left(1 - \frac{R^3}{r_{\rm s}^3}\right) \tag{77}$$

$$\partial_{r_s}\psi|_{r_{\rm s}=R} = -\frac{3}{2}UR\sin^2\vartheta \tag{78}$$

but also

$$\psi = \frac{1}{2}Ar_{\rm s}^2\sin^2\vartheta(R^2 - r_{\rm s}^2) \tag{79}$$

$$\partial_{r_s}\psi|_{r_{\rm s}=R} = -AR^3\sin^2\vartheta \tag{80}$$

such that $\frac{3}{2}\frac{U}{R^2} = A$ and thus $\omega = -\frac{15}{2}\frac{U}{R^2}r$.

4 Further topics

- viscosity and boundary layers create vorticity (e.g. in wakes of obstacles)
- compressible non-barotropic fluids lead to a non-viscous vorticity generation mechanism
- non-conservative forces (Coriolis, ...) create vorticity
- instabilities (in particular shear layers) make a redistribution of vorticity in large-scale structures visible
- Magnus effect as well as the physics of lift involve circulation (wing tip vortex)

5 Literature

- Lamb, *Hydrodynamics*, 1932; §§31–58, §§145–164
- Guyon, *Physical Hydrodynamics*, 2016; Chapter 7