The Rosetta Stone of Index Notation

Wasilij Barsukow
mail@sturzhang.de

February 2015

Version: February 25, 2015

Contents

1 Upstairs or downstairs?

1.1 A word on history

True power must have had the one who made all of the world's mathematicians give their vectors *upper* indices. Quoting from Penrose's "Road to Reality": "Do not be confused here. These are not supposed to be different powers of a single quantity [...]. Confused readers are indeed justified in their confusion. I myself find it not only confusing but also, on occasion, genuinely irritating. For some historical reason, the standard conventions for classical tensor analysis [...] have turned out this way around. These conventions involve tightly-knit rules governing the up/down placing of indices, and the consistent placing for the indices on the coordinates themselves has come out to be in the upper position. (These rules actually work well in practice, but it seems a great pity that the conventions had not been chosen the opposite way around. I am afraid that this is just something that we have to live with.)"

1.2 Introducing indices by not having them

A vector is a quantity not related to any indices. Some might imagine an arrow, other a derivation[1], yet another might think of it as an equivalence class of paths, but all know quite well how to work with them in terms of algebra. So let us give the vector for the moment the nice picture of an arrow ($\nearrow$). You have a good geometric feeling of how to add vectors, and how to multiply them by real numbers, e.g. $2\cdot \rightarrow = \longrightarrow$. That's all you need for a vector space.

Now you can assign every vector a number. Let's call this map ω. For example let's have $\omega(\uparrow) = 5$ and $\omega(\rightarrow) = 7$. You can assign to $\leftarrow$ whatever value you like. But how about making ω a linear map? Then clearly $\omega(\leftarrow) = -7$ and $\omega(\nearrow) = 12$. In fact having fixed the value of ω for two (linearly independent) vectors in 2-d, by linearity you know its value for *any* vector in 2-d.

If you understand this, congratulations! You understand the concept of a **basis** (this was $\uparrow$ and $\rightarrow$) and you understand what a **1-form** is (that was ω – any linear map from vectors to numbers is called a 1-form)!

Of course it might get a bit nasty to draw arrows all the time. So for vectors, let's take the letters u and v, but keep thinking of arrows. Then the linearity of ω expresses itself as

$$\omega(\alpha u + \beta v) = \alpha\omega(u) + \beta\omega(v) \text{ with } \alpha \text{ and } \beta \text{ real numbers}$$

By the way some people like to have vectors written boldface, others put a bar under or an arrow over the letters. I dislike all of that because it is mostly redundant and obstructs the view.

What we have been doing is so-called index-free notation. It is extremely useful and beloved. But nonetheless sometimes we have to condescend to hard work of expliciting a formula in a given basis.

1.3 From lack of letters to abstract nonsense

1.3.1 Vectors

There comes a point in everybody's life when one has to choose some basis. The nice thing is that you often don't need a special basis, you just have to pick *any*. And then you just don't want to waste precious letters for calling every single basis vector differently. Some distinction has to be made, and so you generally would go and index them, e.g. as $e_1 = \rightarrow$, $e_2 = \uparrow$.

Now, for any vector v you know that you can expand it in this basis, e.g. as $v = \alpha e_1 + \beta e_2$. Surely, for the expansion coefficients letters are as precious, and surely this is actually a trivial formula that you would like to have abbreviated (imagine how little less information you get for the enormous amount of extra work of writing this out in 10-d and giving every component an own name). So we will index them as well. It is natural to keep the same letter as the vector (v). You *might* want to write something like $v = \sum_i v_i e_i$ with $v_1 = \alpha$ and $v_2 = \beta$. But you won't. Let us, just because everyone else does, write the components with an upper index. As time goes by you will find a lot of circumstances when this is extremely useful! So we write

$$v = \sum_i v^i e_i$$

where again, every e_i is a vector and every v^i is a number, i.e. the projection of v on e_i.

You see that actually I have not made explicit what the range of summation is. Of course, in a d-dimensional space you are to sum over d such products. Normally however your space does not switch

[1] A derivation is the algebraic formalization of a derivative – a linear map in a suitable space that fulfills the Leibniz rule (the product rule of differentiation).

dimension all the time, so normally you *know* the range. As the last *coup*, let's drop the summation sign – that's called **Einstein notation**, and thus surely must be something clever. By the way, this notation is the only one used, and if Einstein sounds like physics to you – mathematicians use this notation as well.

The rule of the Einstein summation convention is: Sum over all upper-lower index pairs! Let's try this: $v^i e_i = v^1 e_1 + v^2 e_2$. Now, as Penrose said, it is quite nasty, if v^2 actually appears as a square as well... Some people might denote the squared length of a vector like this. You just have to forbid them any such confusing stuff!..

As for any other summation index, the actual name of the index is irrelevant, so $v^i e_i$ is exactly the same as $v^j e_j$. Later, in calculations many different summations get entangled, and the only thing to worry about is not to give different summations the same index.

What have we learned: We use upper indices to denote components of a vector with respect to some basis (which we have to agree upon in advance of course), and lower indices have so far appeared when we were counting our basis vectors. But this is actually not their typical *raison d'être*.

1.3.2 Basis for linear maps

We have written out the linearity of ω before, here it is again:

$$\omega(\alpha u + \beta v) = \alpha\omega(u) + \beta\omega(v)$$

Let's replace u and v by our basis vectors and α and β by components of a vector v. Linearity of ω expresses itself then as

$$\omega(v) = \omega(v^i e_i) = \omega(v^1 e_1 + v^2 e_2) = v^1\omega(e_1) + v^2\omega(e_2) = v^i\omega(e_i)$$

Later, when you get used to this notation you will pass from the first equality sign directly to the last one, without even thinking about it.

These linear maps taking vectors to numbers, again – 1-forms they are called –, form themselves a vector space, i.e. you can add them and multiply by real numbers. If the space where our vectors have been living is called V, then the 1-forms live in its **dual space** V^*. That's just in case you cross the name[2].

What is a clever basis for the 1-forms then? In principle you are free to choose *any*. But there is one special, called the **dual basis** (not a very creative name). It turns out that for every basis vector there exists a 1-form, that takes him to 1, and all the other basis vectors to zero. If you are in d-dimensional space, you have d basis vectors and you will get d such 1-forms. It turns out that they form a basis for the space of 1-forms and this basis is the one called dual. For all times from now on for the 1-forms thou shalt have no other basis before that one!

Can we express these notions already in the new language with indices? Let's take k-th basis vector e_k (index down, right?) and let's give a basis 1-form the generic letter ω again and an *upper* index. Sure, upper indices were given to vector components already, but they are also used to denote the basis 1-forms. Actually, or conversely, lower indices, so far used to denote the basis vectors, will be used for components of 1-forms. That's how life is, and I promise you that there are no more confusions waiting for you. If you get this part, you get everything.

The definition of the k-th 1-form was that it sends the k-th basis vector to 1 and the others to zero. This might remind you of a structure called the Kronecker-delta. Indeed, it is hiding right next

[2]In the context of manifolds you would speak of the tangent and co-tangent spaces.

corner. Let us first write that down:

$$\omega^k(e_k) = 1 \quad \text{no summation over } k \text{ here!} \qquad\qquad \omega^k(e_j) = 0 \,\forall j \neq k$$

You know that you can simplify notation with the Kronecker here. It's just about how to make it formally correct, such that indices work out. You might have seen it in the form

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \qquad\qquad \text{often seen, but wrong}$$

Sorry to disappoint you – here the indices are wrong. The correct δ has its indices like this:

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$$

You will see some chapters later that δ^i_j has one special thing about how exactly the indices are placed and that δ^i_j is actually the i-j-component of the identity matrix. This latter fact will justify why I say that this is the right way of positioning the indices. Now clearly the above definition of the dual basis reads

$$\omega^k(e_j) = \delta^k_j$$

You see that here the indices work out nicely and the nasty note saying that in one of the cases summation is not to be performed is obsolete.

Now that we know that there is a special, nice basis for 1-forms and that the elements of the basis get upper indices, we are ready to dive into true index juggling.

1.3.3 Linear maps

Take a generic 1-form b (the letter ω will from now on be reserved for the basis). Its components with respect to the basis ω^1, ω^2 of 1-forms *must* have *lower* indices for the Einstein summation to work out. Look:

$$b = b_1\omega^1 + b_2\omega^2 = b_i\omega^i$$

Again, every ω^i is a 1-form and every b_i is a number.

In true life, you won't come across the basis vectors or the basis 1-forms very often. Typically a lower index will be the component of a 1-form for you. But it is absolutely necessary to be aware of the double meaning, and to understand its necessity in terms of Einstein notation on one hand, and to be slightly uneasy about having given different objects the same notation on the other.

Time has come for the first true application of all we have learned. Take a 1-form. Let it eat a vector. You get a number. Calculate this number in terms of *components* of the 1-form and of the vector!

$$b = b_i\omega^i \text{ a 1-form} \tag{1}$$

$$v = v^i e_i \text{ a vector} \tag{2}$$

$$b(v) = b(v^i e_i) = v^i b(e_i) = v^i b_j \omega^j(e_i) = v^i b_j \delta^j_i = v^i b_i \tag{3}$$

Isn't that marvelous? You see how all works out automatically on a symbolic level[3].

Perhaps the single new thing is the sum $b_j \delta_i^j$, but I leave it to you to think about it. If you go through the j-summation in your head you will agree that b_i is the correct result.

Take a breath. If you feel that you had enough input for today – it's fine. You have learned everything you need to start using the notation. If you want to go on, you will see what you obtain if you use it.

[3] Also called abstract nonsense.

2 What differential geometry handles

2.1 Linear maps (cont.)

2.1.1 Tensors

There exist linear maps that eat not just one vector, but several. They are called tensors (the word forms, like in 2-forms and so on is reserved for a special subclass). Consider a tensor g eating two vectors and giving a number. Do we have a natural way of finding its components? Yes! With a basis for vectors and for 1-forms there is a natural extension to a basis of all kinds of tensors. An element of the basis for linear maps from two vectors to numbers will exist for every pair of basis vectors, and give 1 if evaluated on them and zero when evaluated on any other pair. This basis will be given the name t for the moment, with its definition being expressed as

$$t^{mn}(e_i, e_j) = \delta_i^m \delta_j^n$$

You see again that t having its indices up is the only way to make this work. Then

$$\begin{aligned} v = v^i e_i \text{ and } u &= u^i e_i \text{ two vectors} \\ g &= g_{ij} t^{ij} \text{ a tensor} \\ g(v,u) = v^i u^j g(e_i, e_j) &= v^i u^j g_{mn} t^{mn}(e_i, e_j) = v^i u^j g_{ij} \end{aligned}$$

Comparing $v^i u^j g(e_i, e_j)$ with the result, one can derive the rule that the component of a tensor in the correspondingly natural basis is just the tensor evaluated with the basis vectors:

$$g(e_i, e_j) = g_{ij}$$

You might also note that

$$t^{mn} = \omega^m \otimes \omega^n$$

Tensors mapping vectors to numbers are called *covariant*, those mapping 1-forms to numbers are called *contravariant*. As there are mixed ones (i.e. those eating one vector and one 1-form, for example), sometimes people say things like *twice covariant*. Our g is such a twice covariant tensor.

2.1.2 Matrices

Two vectors being mapped to a number might remind you of matrices. But a covariant tensor like above is not expressible as a matrix. This is why there is a special section on this topic. Let us first introduce the concept of partial evaluation.

Partial evaluation g, as introduced above, eats two vectors and gives a number. If you give it just one vector, you obtain a structure, that still needs a vector to give a number. An object that needs *one* vector and gives a number is a 1-form! In symbols people would write, if v is to be a vector, that $g(v, \cdot)$ is a 1-form. This is called partial evaluation. You could thus interprete g as something that takes a vector and gives you a 1-form.

Matrices are objects that take a vector and give you a vector back! They thus cannot be twice covariant tensors, like g. They turn out to be mixed tensors. Let's take such an object M and endow it with two slots – one for a vector and one for a 1-form. Call this 1-form b and evaluate M partially with it: $M(b, \cdot)$ is an object that takes a vector and gives a number, thus it is a 1-form again. The components of b are b_i, therefore M's component must have an *upper* index i. For the partial evaluation to be a 1-form, its index must be a lower one. All together, with v a vector, the evaluation $M(b, v)$ is, written in terms of components, the double sum $M^i{}_j v^j b_i$. One index up, one index down, this is why it is called a mixed tensor. You easily find the basis for such tensors to be $\omega^j \otimes e_i$ and the components being given by $M^i{}_j = M(b^i, e_j)$.

Matrix multiplication I suppose you know how to multiply matrices with vectors. In order for the row-times-column-rule to work out you just have to identify the first index in $M^i{}_j$ with the row and the second with the column. Then you have to identify vectors with **column vectors**, in 2-d for example:

$$M^i{}_j v^j = \begin{pmatrix} M^1{}_1 & M^1{}_2 \\ M^2{}_1 & M^2{}_2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} M^1{}_1 v^1 + M^1{}_2 v^2 \\ M^2{}_1 v^1 + M^2{}_2 v^2 \end{pmatrix}$$

Note that we have calculated a partial evaluation $M(\cdot, v)$, which is an object that eats a 1-form and gives a number, and we know that we have calculated a vector! So actually the identification of vectors as atomic quantities and 1-forms as linear operators on them can be completely reversed by taking 1-forms to be the principal objects and vectors to be maps from 1-forms to numbers. By the way you see in the index notation $M^i{}_j v^j$ that the result is a vector, because the free index i is upstairs and thus it must be the component of a vector.

Where does the 1-form enter? Lower indices indexing the column, clearly a 1-form must be identified with a **row vector**. Then the usual law of multiplication for both

$$b_i v^i = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v^1 b_1 + v^2 b_2$$

and

$$b_i M^i{}_j v^j = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} M^1{}_1 & M^1{}_2 \\ M^2{}_1 & M^2{}_2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

works out. Some people say that 1-forms *are* row vectors. This can be misleading, because it makes you think that if you take a vector and write it in a row instead of a column, you get "its" 1-form. This is something that is not true in general, though the concept of "its" 1-form is a very important one. It is introduced in the Section on the musical isomorphism later on. The correct reading of the identification here is that in order to preserve the usual row-times-column-rule of multiplication, you have to write the components of 1-form in a row and those a vector in a column, no more, no less.

Eigenvectors An important thing about matrices is their diagonalizability. The eigenvalues and vectors of course do not depend on notation, but you might wonder how you would write it. Unfortunately you cannot do as much as you might want. The best you get is this

$$M^i{}_j v^j = \Lambda^i{}_j v^j \qquad \text{with } \Lambda \text{ diagonal}$$

You cannot make explicit the eigenvalue itself with $\Lambda = \text{diag}(\lambda_1, \ldots ..., \lambda_d)$, because you would need to write something like

$$M^i{}_j v^j = \lambda_j v^j \quad \text{or} \quad M^i{}_j v^j = \lambda^j v^j \qquad \text{(both wrong)}$$

but clearly, indices do not work out. In principal this is good, it shows you that the collection λ^j or λ_j of eigenvalues is *not* a vector or a 1-form, though it has just one index. It is important to realize that indices is not everything. There exist a lot of quantities that might be put together and indexed – there is no obligation for them to form a proper differential geometric quantity afterwards. This is very true for derivatives, and it is also true for eigenvalues. However you might also see this as a limitation of index notation.

Position of the indices You might have been wondering about the funny way of positioning indices. Why do we write ${M^i}_j$ and not M^i_j? It will become important, once we will have introduced the musical isomorphism. Indices are identified by their position from the left, and not by their up/down position. You can imagine behind each symbol a number of slots for the letter of the index, where every index may be an upper or a lower one, for example something like ${{{T^{ij}}_k}^m}_{sr}$. For later usage, get used already not to "fill up" empty spaces both upstairs and downstairs by shifting indices to the left, like in T^{ijm}_{ksr}. Such a notation does not have an interpretation in differential geometry. The former however designates a tensor, whose first two and the fourth slots eat a 1-form and the others eat a vector. As the order matters, you definitely have to know which object goes in which slot!

The only exception to this rule is the identity matrix δ^i_j, which traditionally is written with no specification about the order of the indices.

2.1.3 Contractions or traces

Any pair of a covariant and a contravariant index may be summed over. Given the matrix ${M^i}_j$, one might consider the scalar ${M^i}_i = {M^0}_0 + \cdots + {M^d}_d$. This is called the **trace** of M, or the contraction of its two indices. For contractions of two covariant, or two contravariant indices one needs the musical isomorphism of the next Section.

2.1.4 Forms

[to be added]

2.2 Metric duality or the musical isomorphism

Recall how a 1-form b is evaluated on a vector v:

$$b(v) = b_i v^i = \begin{pmatrix} b_1 & \cdots & b_d \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}$$

This looks very much like a scalar product, but is not a scalar product because it does not involve two vectors. One way to think of what follows is that one needs to convert a 1-form into a vector. It might seem a trivial thing to write what has been a column vector into a row, but actually we are identifying a vector with some 1-form via components. So maybe it is worth studying the true mathematical structure of this operation. Even if this ought to be a bijection, surely there is in principle not a unique way of identification, and one might wonder which one to prefer. The other approach to this problem is that a scalar product $\langle v, u\rangle$ is not necessarily just the sum $v^1u^1 + \cdots + v^d u^d$, but in general is given by the Hermitian form

$$\langle v, u\rangle = v^{\mathrm{T}} M u$$

with M a symmetric positive definite matrix.

A matrix is not precisely what we want (that's the reason there is a transposition in the formula above). A matrix is a map from vectors to vectors, and thus not the right thing. We need a map from 1-forms to vectors, or from vectors to 1-forms. We have encountered such an object already, a twice covariant tensor. It maps two vectors to a number; if you feed it just one vector is becomes a 1-form. Thus a twice covariant tensor is a map from vectors to 1-forms. Analogously a twice contravariant tensor is a map from 1-forms to vectors. Let us thus define

$$\langle v, u \rangle = g(v, u) = g_{ij} v^i u^j$$

This is at least the index-correct version of the equation before. But it is more.

Scalar products tell you something about lengths and angles. You would not be surprised that the tensor g is called the **metric tensor**. It can be used to measure true lengths (on a curved manifold, say). This would however lead us away from tensor algebra to true differential geometry, so let's not go into more details on what you can do with it otherwise. For us it shall just be the object that we need to convert 1-forms to vectors. By partial evaluation, $g(v, \cdot)$ is a 1-form and by fixing an invertible g we fix an isomorphism between the space of vectors and the space of 1-forms. In brief, if g is symmetric and positive definite (indefinite), the isomorphism leads to what is called **(semi-)Riemannian geometry**, an antisymmetric g is used to define a **symplectic structure**.

In index notation, one easily finds the components of $g(v, \cdot)$ to be $g_{ij} v^i$. This one form is a one-to-one map of v, so it is common to use the same letter for it:

$$v_j = g_{ij} v^i$$

The metric tensor thus allows to "pull indices up and down", or provides a bijection between 1-forms and vectors. This bijection is called **metric duality** and $g(v, \cdot)$ the **metric dual** of v.

There is a way of expressing the metric dual with the same letter (i.e. without the construction $g(v, \cdot)$) in an index-free manner. If v is a vector, then its metric dual is denoted by $v^\flat$ and if b is a 1-form its metric dual is denoted by $b^\sharp$. This makes allusion to a vector lowering its index and a $\flat$ (**flat**) lowering a musical note (by a half-tone), and analogously for $\sharp$ (**sharp**). Some people call the metric duality therefore the **musical isomorphism**.

One can use g to rise or lower indices of tensors as well, e.g. $T_{ij} = g_{ik} T^k{}_j = g_{ik} g_{jm} T^{km}$. The inverse of g must have two upper indices for the summations to work out, and it will be given the same letter:

$$g_{ij} g^{jk} = \delta_i^k$$

Contractions of index pairs that are not mixed become possible now, by raising, or lowering one of them. For a twice covariant tensor T for example

$$\operatorname{tr} T := \operatorname{tr} g^{ij} T_{jk} = g^{kj} T_{jk}$$