A low-Mach Roe-type solver for the Euler equations allowing for gravity source terms

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A time-continuous finite volume scheme for a system of conservation laws $\partial_t q + \nabla \cdot f(q) = 0$ with $q : \mathbb{R}^d \times \mathbb{R}^+_0 \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ reads (e.g. in $d = 2$):

$$\partial_t q_i + \frac{f^{(x)}_{i+\frac12,j} - f^{(x)}_{i-\frac12,j}}{\Delta x} + \frac{f^{(y)}_{i,j+\frac12} - f^{(y)}_{i,j-\frac12}}{\Delta y} = 0$$

Roe-type schemes:

$$f^{(x)}_{i+\frac12} = \frac12 (f(q_{i+1}) + f(q_i)) - \frac12 D_{i+\frac12}(q_{i+1} - q_i)$$

E.g. **Roe-scheme**: $D_{i+\frac12} = |f'|$ evaluated at $\langle q \rangle_{i+\frac12}$

Incompressible limit

For the Euler equations in d spatial dimensions

$$n = d+2 \qquad q = (\rho,\, \rho v,\, e)^{\mathrm{T}} \qquad f = \left(\rho v,\, \rho v \otimes v + \frac{p}{\epsilon^2},\, v(e+p)\right)^{\mathrm{T}}$$

together with $e = \frac{p}{\gamma - 1} + \frac12 \epsilon^2 \rho |v|^2$ and the local Mach number $M = \frac{v}{\sqrt{\gamma p/\rho}} \sim \epsilon$.

Expand quantities as power series in ϵ, e.g.

$$p = p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \dots$$

Limit in the continuous case: **incompressible hydrodynamics** with $p^{(2)}$ as the dynamic pressure, as well as $\boxed{\nabla p^{(0)} = \nabla p^{(1)} = 0}$.

Modification of the diffusion matrix

Roe matrix:

$$D_{\mathrm{Roe}} \in \left(\mathcal{O}\left(\tfrac{1}{\epsilon}\right) \; \mathcal{O}\left(\tfrac{1}{\epsilon}\right) \;\middle|\; \begin{matrix} \mathcal{O}\left(\frac{1}{\epsilon}\right) \\ \mathcal{O}\left(\frac{1}{\epsilon}\right) \\ \mathcal{O}\left(\frac{1}{\epsilon}\right) \end{matrix} \right)$$

terms violating asymptotics

Take $\boxed{D = P^{-1}|Pf'|}$ and use P to finetune scalings of D. Weiss & Smith 95 / Turkel 99 gives

$$D_{\mathrm{WS\text{-}T}} \in \left(\mathcal{O}(1) \;\middle|\; \begin{matrix} \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \\ \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \\ \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \end{matrix} \right)$$

We suggest (Miczek+ 15, Barsukow+ in prep.) a P such that

$$D \in \left(\mathcal{O}(1) \;\middle|\; \begin{matrix} \mathcal{O}(1) \\ \frac{\gamma-1}{\epsilon^2} \\ \mathcal{O}(1) \end{matrix} \right)$$

For supersonic flows we recover the Roe scheme in a continuous manner.

Formally:

$$p^{(0)}_{i+1} - p^{(0)}_{i-1} = 0$$
$$p^{(1)}_{i+1} - p^{(1)}_{i-1} = \Delta x(\dots) + \mathcal{O}(\Delta x^2)$$

Formally:

$$\boxed{p_i^{(\ell)} - p_{i-1}^{(\ell)} = 0 \quad \longrightarrow \quad \nabla p^{(\ell)} = 0}$$

Discrete equation reproduces the analytic constraint even for finite Δx! $(\ell = 0, 1)$

Kinetic energy

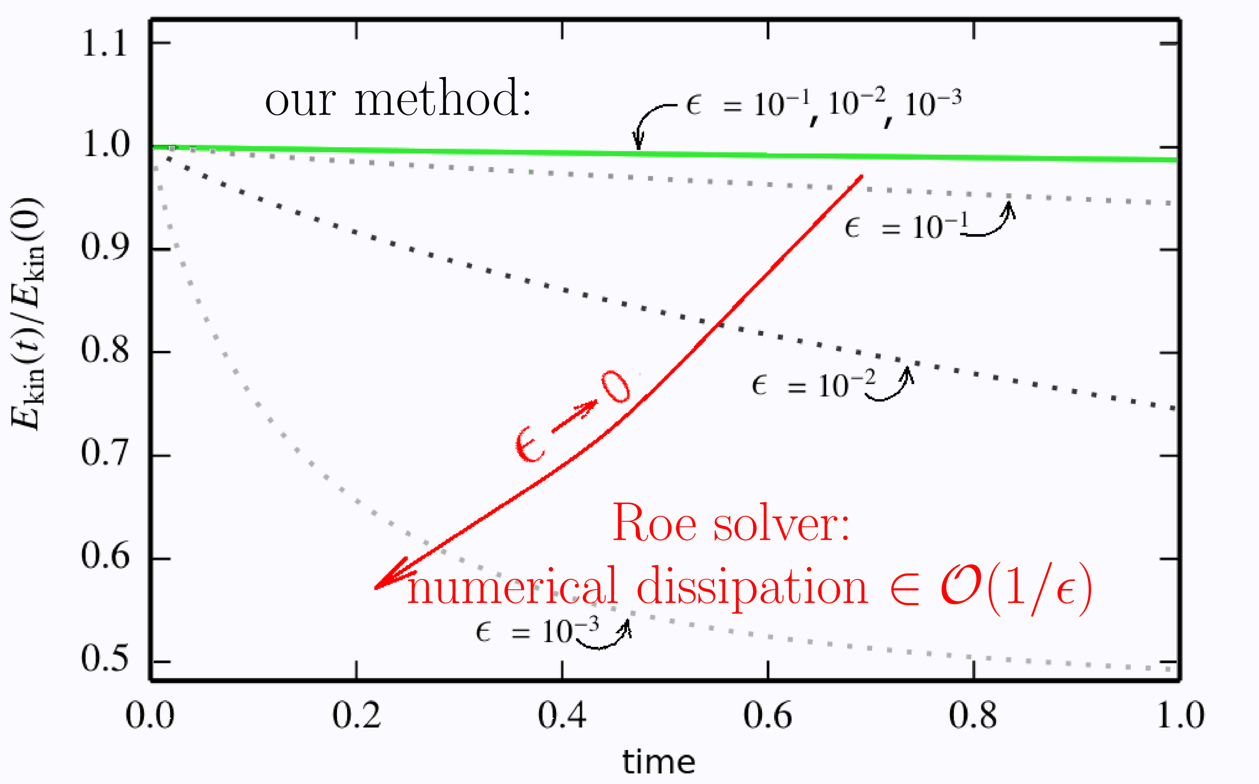
The equation for the kinetic energy $e_{\mathrm{kin}} = \frac{\rho |v|^2}{2}$ can be written as

$$\partial_t e_{\mathrm{kin}} + \nabla \cdot \left(v \left(e_{\mathrm{kin}} + \frac{p}{\epsilon^2} \right) \right) = \frac{p}{\epsilon^2} \nabla \cdot v \quad \notin \mathcal{O}(\epsilon).$$

and is equivalent to

$$\partial_t e_{\mathrm{kin}} + \nabla \cdot \left(v \left(e_{\mathrm{kin}} + p^{(2)} \right) \right) + \mathcal{O}(\epsilon) = p^{(2)} \nabla \cdot v \quad \in \mathcal{O}(\epsilon).$$

Kinetic energy is a conserved quantity in the limit $\epsilon \to 0$.



Time integration

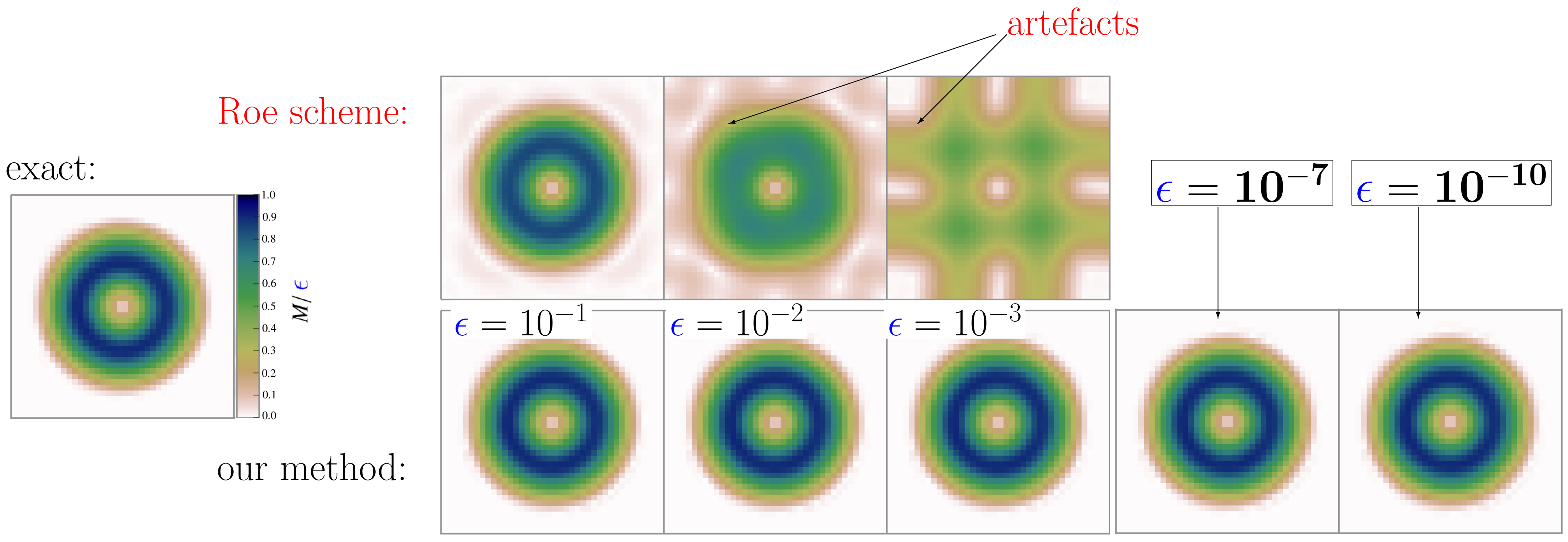
\# **explicit time integration**:

- linear von Neumann stability can be performed completely due to decomposing eigenspace of the amplification matrix
- CFL constraint $\frac{\Delta t}{\Delta x} \in \mathcal{O}(\epsilon^2)$

\# **implicit time integration**: necessary for near-incompressible flow to overcome separation of acoustic and advective time scales!

- for accuracy: advective time step $\frac{\Delta t}{\Delta x} \sim \frac{1}{v} \in \mathcal{O}(1)$
- ESDIRK-schemes and the Newton-Raphson method for the implicit steps
- (preconditioned) iterative algorithms for the linear systems
- Computation and storage of the Jacobian (sparse with dense blocks) important issues for an efficient implementation

Example: stationary, incompressible 2-d vortex



Influence of gravity source terms

With Ma/Fr = 1, Ma = $\epsilon \to 0$

$$\partial_t \begin{pmatrix} \rho \\ \rho v \\ e \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho v \\ \rho v \otimes v + \frac{p}{\epsilon^2} \\ v(e+p) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\epsilon^2} \rho g \\ \rho g \cdot v \end{pmatrix}$$

Its limit are hydrostatic equilibria with $\boxed{\nabla p^{(0)} = \rho^{(0)} g^{(0)}}$.

Energy as time-continuous scheme for Weiss & Smith 95 / Turkel 99 (bottom row of matrix $D_{\mathrm{WS\text{-}T}}$):

$$\partial_t e + \frac{1}{\Delta x} \underbrace{(\text{central flux})}_{\in \mathcal{O}(1)} + \frac{1}{\Delta x} \underbrace{(\text{diffusive part})}_{\in \mathcal{O}(1/\epsilon^2)} = \underbrace{\rho g \cdot v}_{\in \mathcal{O}(1)}$$

The highest order equation (formally) would still impose $\nabla p^{(0)} = 0$ – the method is thus not asymptotic preserving.

The new modification overcomes this problem:

Our diffusion matrix D does not have entries proportional to $\frac{1}{\epsilon^2}$ in its energy row!

References:

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Miczek, F., Röpke, F. K., Edelmann, P. V. F. 2015, Astronomy & Astrophysics, 576, A50

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