Low Mach number asymptotics

↘M# seminar, U Würzburg
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Overview

- Some examples
- Scalings, families of solutions, reference quantities, rescaled equations and limits
- Looking at the limit of modified equations
- Discretizations of a limit model

Mach number?

The equations of compressible hydrodynamics

$$\partial_t \varrho + \partial_i(\varrho v^i) = 0 \tag{1}$$

$$\partial_t(\varrho v^j) + \partial_i(\varrho v^i v^j) + \delta^{ij}\partial_i p = 0 \tag{2}$$

$$\partial_t e + \partial_i(v^i(e+p)) = 0 \tag{3}$$

with the *equation of state* $e = \dfrac{p}{\gamma - 1} + \dfrac{1}{2}\varrho|v|^2$, $\gamma > 1$ admit approximate solutions

$$\varrho = \bar{\varrho} + \hat{\varrho} \qquad v^i = \hat{v}^i \qquad p = \bar{p} + \hat{p} \tag{4}$$

with $\hat{\varrho} \ll \bar{\varrho} = \text{const}$, and so on, and any derivatives or divergences of the same order as the quantity. Every disturbance then satisfies the wave equation, e.g. $\hat{\varrho}$:

$$\partial_t^2 \hat{\varrho} - c^2 \partial^i \partial_i \hat{\varrho} = 0 \tag{5}$$

with c given by the background: $c = \sqrt{\dfrac{\gamma \bar{p}}{\bar{\varrho}}}$.

Mach number?

The local Mach number is the dimensionless quantity

$$M(x,t) := \frac{|v(x,t)|}{\sqrt{\frac{\gamma p(x,t)}{\varrho(x,t}}} : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^+ \tag{6}$$

whose value you can evaluate at any point of the flow without reference to the actual presence of sound waves or the fulfillment of the above assumptions (in the spirit: if there were a sufficiently weak and short-wavelength sound wave there, what would be its speed?)

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There exists also something called *reference Mach number*, which will not be used in this talk.

Example – sound waves

$$\varrho = \bar{\varrho}(1 + M\cos(-\omega t + kx)) \tag{7}$$

$$v = Mc\cos(-\omega t + kx) \tag{8}$$

$$e = \bar{e} + \frac{\bar{\varrho}c^2}{\gamma - 1} M\cos(-\omega t + kx) \tag{9}$$

$$P = \bar{P} + \bar{\varrho}c^2 M\cos(-\omega t + kx) \tag{10}$$

with $c = \frac{\omega}{k} = \left(\gamma \frac{\bar{P}}{\bar{\varrho}}\right)^{1/2}$ and the representative Mach number $M \in \mathbb{R}^+$ having been chosen as the local Mach number of the wave peak.

Example – stationary 2D vortex

All functions are functions of $r := (x^2 + y^2)^{1/2}$ only, and

$$v = v^\phi(r) e_\phi \tag{11}$$

$$p = \int_0^r \mathrm{d}r' \varrho(r') \frac{v^\phi(r')^2}{r'} + p(r = 0) \tag{12}$$

with $\varrho(r)$ and $v^\phi(r)$ any pair of functions leading to a sensible value of the above integral.

Possible choice of representative Mach number: value of the local Mach number at the maximum of v^ϕ, or its average over the domain, or...

Low Mach asymptotics?

What happens when $M \to 0$?

Choice of scalings

A family of 2D vortices, parametrized by M:

$$v = Mv^{\phi}(r)e_{\phi} \tag{13}$$

$$p = \int_0^r \mathrm{d}r' \varrho(r') M^2 \frac{v^{\phi}(r')^2}{r'} + p(r=0) \tag{14}$$

Another, very similar family of such vortices:

$$v = v^{\phi}(r)e_{\phi} \tag{15}$$

$$p = \int_0^r \mathrm{d}r' \varrho(r') \frac{v^{\phi}(r')^2}{r'} + \frac{p(r=0)}{M^2} \tag{16}$$

And yet another one:

$$v = \sqrt{M}v^{\phi}(r)e_{\phi} \tag{17}$$

$$p = \int_0^r \mathrm{d}r' \varrho(r') M^{3/2} \frac{v^{\phi}(r')^2}{r'} + p(r=0) \tag{18}$$

$$\text{and scale the density as } \sqrt{M}\varrho. \tag{19}$$

In all three cases the local Mach number will be proportional to M!

Choice of scalings

Hidden scalings:

$$\varrho = \bar{\varrho}(1 + M\cos(-M^{1000}\omega t + M^{1000}kx)) \quad (20)$$

$$v = Mc\cos(-M^{1000}\omega t + M^{1000}kx) \quad (21)$$

$$e = \bar{e} + \frac{\bar{\varrho}c^2}{\gamma - 1}M\cos(-M^{1000}\omega t + M^{1000}kx) \quad (22)$$

$$P = \bar{P} + \bar{\varrho}c^2 M\cos(-M^{1000}\omega t + M^{1000}kx) \quad (23)$$

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time scale t^* must be propotional to $\frac{x^*}{v^*}$;
pressure p^* and energy scale e^* proportional to $\varrho^*(v^*)^2$.

Now you would want to make them depend on M – but M is dimensionless... So you are free to do whatever you like!

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If you assume that the nondimensional quantities are $\mathcal{O}(1)$, then you must fulfill

$$M \sim \frac{v^*}{\sqrt{p^*/\varrho^*}} \tag{24}$$

This is not a restriction, if the expansion of your original quantity has a terminating Laurent series (i.e. a minimal power exists).

Rescaled equation

Take in the following

$$x = M^{\mathfrak{a}} x^* \tilde{x} \qquad t = M^{\mathfrak{b}} \frac{x^*}{v^*} \tilde{t} \tag{25}$$

$$\varrho(x,t) = M^{\mathfrak{c}} \varrho^* \cdot \tilde{\varrho}(\tilde{x},\tilde{t}) \qquad v(x,t) = M^{\mathfrak{d}} v^* \cdot \tilde{v}(\tilde{x},\tilde{t}) \tag{26}$$

$$e(x,t) = M^{\mathfrak{e}} \varrho^* (v^*)^2 \cdot \tilde{e}(\tilde{x},\tilde{t}) \qquad p(x,t) = M^{\mathfrak{f}} \varrho^* (v^*)^2 \cdot \tilde{p}(\tilde{x},\tilde{t}) \tag{27}$$

Non-dimensional quantities being asymptotically constant ($\mathcal{O}(1)$) and M having the interpretation of a Mach number, one needs

$$M \sim \frac{v}{\sqrt{p/\varrho}} = \underbrace{\frac{v^* \tilde{v} \cdot \sqrt{\varrho^* \tilde{\varrho}}}{\sqrt{\varrho^* (v^*)^2 \cdot \tilde{p}}}}_{\mathcal{O}(1)} \frac{M^{\mathfrak{d}} \sqrt{M^{\mathfrak{c}}}}{\sqrt{M^{\mathfrak{f}}}} \qquad \Rightarrow \qquad \mathfrak{c} + 2\mathfrak{d} = \mathfrak{f} + 2 \tag{28}$$

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Additionally, via the equation of state the e and p scalings are linked:

$$e = \frac{p}{\gamma - 1} + \frac{1}{2} \varrho |v|^2 \tag{29}$$

$$M^{\mathfrak{e}} \varrho^* (v^*)^2 \cdot \tilde{e} = \frac{M^{\mathfrak{f}} \varrho^* (v^*)^2 \cdot \tilde{p}}{\gamma - 1} + \frac{1}{2} M^{\mathfrak{c}} \varrho^* \tilde{\varrho} \cdot M^{2\mathfrak{d}} (v^*)^2 \cdot |\tilde{v}|^2 \tag{30}$$

$$\Rightarrow \mathfrak{e} = \min(\mathfrak{f}, \mathfrak{c} + 2\mathfrak{d}) = \min(\mathfrak{f}, \mathfrak{f} + 2) = \mathfrak{f} \tag{31}$$

Rescaled equation

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$$\varrho(x,t) = M^{\mathfrak{f}+2-2\mathfrak{d}} \varrho^* \cdot \tilde{\varrho}(\tilde{x},\tilde{t}) \qquad v(x,t) = M^{\mathfrak{d}} v^* \cdot \tilde{v}(\tilde{x},\tilde{t}) \tag{33}$$

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Rewriting the Euler equations now gives:

$$M^{\mathfrak{a}-\mathfrak{d}-\mathfrak{b}} \partial_{\tilde{t}} \tilde{\varrho} + \tilde{\partial}_i (\tilde{\varrho} \tilde{v}^i) = 0 \tag{35}$$

$$M^{\mathfrak{a}-\mathfrak{d}-\mathfrak{b}} \partial_{\tilde{t}} (\tilde{\varrho} \tilde{v}^j) + \tilde{\partial}_i (\tilde{\varrho} \tilde{v}^i \tilde{v}^j) + \frac{1}{M^2} \delta^{ij} \tilde{\partial}_i \tilde{p} = 0 \tag{36}$$

$$M^{\mathfrak{a}-\mathfrak{d}-\mathfrak{b}} \partial_{\tilde{t}} \tilde{e} + \tilde{\partial}_i (\tilde{v}^i (\tilde{e} + \tilde{p})) = 0 \tag{37}$$

Equation of state:

$$\tilde{e} = \frac{\tilde{p}}{\gamma - 1} + \frac{1}{2} M^2 \tilde{\varrho} \cdot |\tilde{v}|^2 \tag{38}$$

Instead of a family of solutions we are left with a family of equations.

Limit of the rescaled equations

Idea: Expand the rescaled quantities as power series in M, e.g.

$$\tilde{\varrho} = \tilde{\varrho}^{(0)} + M\tilde{\varrho}^{(1)} + M^2\tilde{\varrho}^{(2)} + \dots \quad \text{such that} \quad \tilde{\varrho}^{(k)} \in \mathcal{O}(1)\,\forall k \tag{39}$$

$$\partial_\cdot \tilde{\varrho} = \partial_\cdot \tilde{\varrho}^{(0)} + M\partial_\cdot \tilde{\varrho}^{(1)} + M^2\partial_\cdot \tilde{\varrho}^{(2)} + \dots \tag{40}$$

Now you would like to insert this into the rescaled equations and collect order by order. Problem: Actually $\tilde{\varrho}^{(k)} \in \mathcal{O}(1) \not\Rightarrow \partial_\cdot \tilde{\varrho}^{(k)} \in \mathcal{O}(1)$. Collecting formally order by order however *does* assume this.

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$$\text{If } \tilde{\varrho} \in \mathcal{O}(1) \quad \Rightarrow \quad \partial_{\cdot}\tilde{\varrho} \in \mathcal{O}(1) \tag{41}$$

$$\text{then } \varrho \in \mathcal{O}(M^{\mathfrak{c}}) \quad \Rightarrow \quad \partial_i\varrho \in \mathcal{O}(M^{\mathfrak{c}+\mathfrak{a}}) \tag{42}$$

$$\partial_t\varrho \in \mathcal{O}(M^{\mathfrak{c}+\mathfrak{b}}) \tag{43}$$

Recall:

$$x = M^{\mathfrak{a}}x^*\tilde{x} \qquad t = M^{\mathfrak{b}}\frac{x^*}{v^*}\tilde{t} \tag{44}$$

$$\varrho(x,t) = M^{\mathfrak{c}}\varrho^* \cdot \tilde{\varrho}(\tilde{x},\tilde{t}) \tag{45}$$

Strouhal number

Rieper 2011

$$M^{\mathfrak{a}-\mathfrak{d}-\mathfrak{b}}\partial_{\tilde{t}}\tilde{\varrho} + \tilde{\partial}_i(\tilde{\varrho}\tilde{v}^i) = 0 \tag{46}$$

$$M^{\mathfrak{a}-\mathfrak{d}-\mathfrak{b}}\partial_{\tilde{t}}(\tilde{\varrho}\tilde{v}^j) + \tilde{\partial}_i(\tilde{\varrho}\tilde{v}^i\tilde{v}^j) + \frac{1}{M^2}\delta^{ij}\tilde{\partial}_i\tilde{p} = 0 \tag{47}$$

$$M^{\mathfrak{a}-\mathfrak{d}-\mathfrak{b}}\partial_{\tilde{t}}\tilde{e} + \tilde{\partial}_i(\tilde{v}^i(\tilde{e}+\tilde{p})) = 0 \tag{48}$$

The combination $\mathfrak{a}-\mathfrak{d}-\mathfrak{b}$ is found in the dimensionless Strouhal number:

$$\mathrm{Str} = \frac{x}{vt} = \frac{M^{\mathfrak{a}}x^*\tilde{x}}{M^{\mathfrak{d}}v^*\tilde{v}\cdot M^{\mathfrak{b}}\frac{x^*}{v^*}\tilde{t}} \in \mathcal{O}(1) \quad \text{if} \quad \mathfrak{a}-\mathfrak{d}-\mathfrak{b} = 0 \tag{49}$$

Incompressible limit

Take $\boxed{\mathfrak{a} = \mathfrak{d} + \mathfrak{b}}$: Klainerman & Majda 1981, Kreiss et al. 1991, Schochet 1994

$$\partial_{\tilde t}\tilde\varrho + \tilde\partial_i(\tilde\varrho\tilde v^i) = 0 \tag{50}$$

$$\partial_{\tilde t}(\tilde\varrho\tilde v^j) + \tilde\partial_i(\tilde\varrho\tilde v^i\tilde v^j) + \frac{1}{M^2}\delta^{ij}\tilde\partial_i\tilde p = 0 \tag{51}$$

$$\partial_{\tilde t}\tilde e + \tilde\partial_i(\tilde v^i(\tilde e + \tilde p)) = 0 \tag{52}$$

$$\tilde e = \frac{\tilde p}{\gamma - 1} + \frac{1}{2}M^2\tilde\varrho\cdot|\tilde v|^2 \tag{53}$$

Expanding every rescaled quantity into a power series in M yields to highest order and assuming spatial constants to be constant in time as well

$$\partial_{\tilde t}\tilde\varrho^{(0)} + \tilde\partial_i(\tilde\varrho^{(0)}\tilde v^{i(0)}) = 0 \quad\Rightarrow\quad \partial_{\tilde t}\tilde\varrho^{(0)} + \tilde v^{i(0)}\tilde\partial_i\tilde\varrho^{(0)} = 0 \tag{54}$$

$$\tilde\partial_i\tilde p^{(0)} = \tilde\partial_i\tilde p^{(1)} = 0 \tag{55}$$

$$\tilde\varrho^{(0)}\left(\partial_{\tilde t}\tilde v^{j(0)} + \tilde v^{i(0)}\tilde\partial_i\tilde v^{j(0)}\right) + \delta^{ij}\tilde\partial_i\tilde p^{(2)} = 0 \tag{56}$$

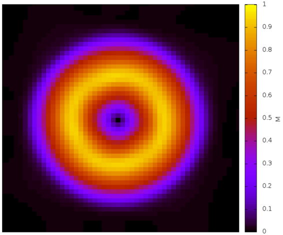
$$\partial_{\tilde t}\tilde e^{(0)} + \tilde\partial_i(\tilde v^{i(0)}(\tilde e^{(0)} + \tilde p^{(0)})) = 0 \quad\Rightarrow\quad \tilde\partial_i\tilde v^{i(0)} = 0 \tag{57}$$

(58)

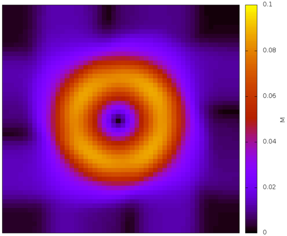
Low Mach number behaviour of compressible codes

Volpe 1993 and many more

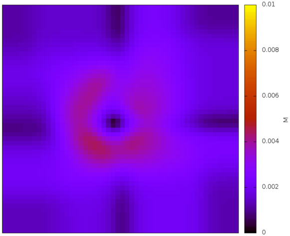
Here: performance of a WENO5 code on an incompressible, stationary, 2D vortex (Gresho) (Mach number plots for $t = 1$)



$M = 10^0$



$M = 10^{-1}$



$M = 10^{-2}$

Wrong pressure fluctuations?

Guillard & Murrone 2004

Choose initial data close to a constant density incompressible flow, i.e. initially at rest and with

$$\tilde{p} = \tilde{p}_0 + M^2 \tilde{p}_2 + \dots \tag{59}$$

both sides of the discontinuity and solve the exact Riemann problem in the limit $M \to 0$ to obtain the pressure between e.g. a 1-rarefaction and a 3-shock:

$$\tilde{p} = \tilde{p}_0 - \frac{M}{2} \sqrt{\gamma \tilde{p}_0 \tilde{\varrho}_0} \cdot [\tilde{v}_0] + \dots \tag{60}$$

Artificial sound waves produced at the discontinuity lead away from an incompressible solution.

Wrong scaling of modified equations?

Dellacherie 2010

$$\partial_t q + \partial_k f^k(q) = 0 \qquad \text{with} \qquad q \text{ the vector of conserved quantities} \tag{61}$$

$$f^k(q) \text{ a vector of fluxes for each } k \tag{62}$$

is replaced by the semi-discrete equation

$$(\partial_t q)_i + \sum_k \frac{(F^k)_{i+\frac{1}{2}} - (F^k)_{i-\frac{1}{2}}}{\Delta x_k} = 0 \tag{63}$$

The Roe solver takes

$$(F^k)_{i+\frac{1}{2}} = \frac{1}{2}\left(f^k(q_i) + f^k(q_{i+1}) - \left| \left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\text{Roe}} \right| (q_{i+1} - q_i) \right) \tag{64}$$

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Dellacherie evaluates

$$\tilde{p}_{i+\frac{1}{2}} := (M^2 \tilde{\varrho}\tilde{v}^2 + p)_{i+\frac{1}{2}} - M^2 (\tilde{\varrho}\tilde{v})_{i+\frac{1}{2}} \cdot \langle \tilde{v}_{i+\frac{1}{2}} \rangle_{\text{Roe}} \tag{66}$$

$$= \frac{\tilde{p}_i + \tilde{p}_{i+1}}{2} - M \left\langle \frac{\tilde{\varrho}\tilde{c}[\tilde{v}]}{2} \right\rangle_{\text{Roe}} - M \left\langle \frac{[\tilde{v}][\tilde{p}]}{2c} \right\rangle_{\text{Roe}} \tag{67}$$

$$= \frac{\tilde{p}_i + \tilde{p}_{i+1}}{2} + \mathcal{O}(M \Delta x) \tag{68}$$

Wrong scaling of modified equations?

Miczek et al. 2014

Consider again the Roe solver:

$$\frac{(F^k)_{i+\frac{1}{2}} - (F^k)_{i-\frac{1}{2}}}{\Delta x} = \frac{f^k(q_{i+1}) - f^k(q_{i-1})}{\Delta x} \tag{69}$$

$$+ \frac{\left| \left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\text{Roe}@i-1,i} \right| (q_i - q_{i-1}) - \left| \left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\text{Roe}@i,i+1} \right| (q_{i+1} - q_i)}{2\Delta x} \tag{70}$$

$$\simeq \frac{\partial f^k}{\partial q}_i + \mathcal{O}(\Delta x^2) + \left| \left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\text{Roe}@i,i+1} \right| \Delta x (\partial_x^2 q)_i + \text{more terms} \tag{71}$$

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Compare

$$\frac{\partial f^k}{\partial q}_i \sim \begin{pmatrix} 0 & \mathcal{O}(1) & 0 \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}\left(\frac{1}{M}\right) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \quad \left|\left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\text{Roe@}i,i+1}\right| \sim \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(M) & \mathcal{O}\left(\frac{1}{M}\right) \\ \mathcal{O}\left(\frac{1}{M}\right) & \mathcal{O}\left(\frac{1}{M}\right) & \mathcal{O}\left(\frac{1}{M}\right) \\ \mathcal{O}(M) & \mathcal{O}(M) & \mathcal{O}\left(\frac{1}{M}\right) \end{pmatrix} \tag{72}$$

Asymptotic preserving

In the limit $M \to 0$ the discrete equations become a valid discretization of the continous limit model.

In other words, the limits $\Delta x \to 0$ and $M \to 0$ commute.

Example – Roe solver

Guillard & Viozat 1999, Rieper 2011, Miczek et al. 2014

$$\begin{aligned}\Delta x \partial_t (\varrho v)_i + \frac{1}{2} \Bigg((\varrho v^2)_{i+1} - (\varrho v^2)_{i-1} + \frac{p_{i+1}}{M^2} - \frac{p_{i-1}}{M^2} \\ + \left\langle -\frac{cv}{M} \right\rangle_{i-1,i} [\varrho]_{i-1,i} + \mathcal{O}(1) + \left\langle \frac{c}{M} \right\rangle_{i-1,i} [\varrho v]_{i-1,i} + \left\langle \frac{2(\gamma-1)v}{cM} \right\rangle_{i-1,i} [e]_{i-1,i} \\ - \left\langle -\frac{cv}{M} \right\rangle_{i,i+1} [\varrho]_{i,i+1} + \mathcal{O}(1) - \left\langle \frac{c}{M} \right\rangle_{i,i+1} [\varrho v]_{i,i+1} - \left\langle \frac{2(\gamma-1)v}{cM} \right\rangle_{i,i+1} [e]_{i,i+1} \Bigg)\end{aligned}$$

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$$+ \left\langle -\frac{cv}{M} \right\rangle_{i-1,i} [\varrho]_{i-1,i} + \mathcal{O}(1) + \left\langle \frac{c}{M} \right\rangle_{i-1,i} [\varrho v]_{i-1,i} + \left\langle \frac{2(\gamma-1)v}{cM} \right\rangle_{i-1,i} [e]_{i-1,i}$$
$$- \left\langle -\frac{cv}{M} \right\rangle_{i,i+1} [\varrho]_{i,i+1} + \mathcal{O}(1) - \left\langle \frac{c}{M} \right\rangle_{i,i+1} [\varrho v]_{i,i+1} - \left\langle \frac{2(\gamma-1)v}{cM} \right\rangle_{i,i+1} [e]_{i,i+1} \Bigg)$$

$$\mathcal{O}\left(\frac{1}{M^2}\right) \; : \; p^{(0)}_{i+1} - p^{(0)}_{i-1} = 0$$
$$\mathcal{O}\left(\frac{1}{M}\right) \; : \; p^{(1)}_{i+1} - p^{(1)}_{i-1} = \langle -cv \rangle_{i-1,i} [\varrho]_{i-1,i} + \langle c \rangle_{i-1,i} [\varrho v]_{i-1,i} + \left\langle \frac{2(\gamma-1)v}{c} \right\rangle_{i-1,i} [e]_{i-1,i}$$
$$- \left\{ \langle -cv \rangle_{i,i+1} [\varrho]_{i,i+1} + \langle c \rangle_{i,i+1} [\varrho v]_{i,i+1} + \left\langle \frac{2(\gamma-1)v}{c} \right\rangle_{i,i+1} [e]_{i,i+1} \right\}$$

Summary & outlook

- Comparing discrete and continous solutions suffers from the fact that you cannot look at every single one
- Looking at the modified equations, if the problems are in the error terms, suffers from the fact that you cannot look at every single one
- Looking order by order at the non-dimensional, discrete equations assumes an expansion and concentrates on one limit out of possibly many different ones (but the amount of work is finite!)

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- Good and bad discretizations of the limit model?
- How do the results of the three methodologies above compare?
- How do IMEX-schemes and asymptotic preserving time discretizations fit in here?
- Checkerboard problem and well-balancing?
- Counter-examples to the expansion in powers of M?