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Cluck-Cluck Effect

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1 Introduction

While emtying a wine bottle one is inevitably confronted with the fact that the fluid does not leave its container at constant rate, but is ejected in seemingly well-defined doses. We call this phenomenon *cluck-cluck effect* for obvious reasons and want to investigate the frequency of the oscillation and the typical volume expelled during one *cluck*. Incompressibility shall be assumed in what follows.

2 Flow inside constant cross section

2.1 Introduction

The situation we are facing here is that of a vertically oriented pipe of radius R everywhere, closed at top and at the bottom and filled with a fluid. At some point in time the bottom is opened. The observations show that in pipes of small diameter nothing happens (i.e. the fluid does not leave the pipe) and in pipes with larger diameter air is entering through the middle and the fluid is flowing down along the walls. It has been observed that the fluid in the top regions not yet reached by the rising air column is practically at rest. Thus this information is propagating with a much smaller speed than sound.

In the case of an unchanging cross section of the container which is being emptied no cluck-cluck has ever been observed.

We assume the velocity of the flow to be constant along its direction. This is a typical situation in which the Navier-Stokes-equations are exactly soluble. We call this solution Poisseuille flow by analogy with the flow through a pipe, although the boundary conditions in our case differ significantly. The flow of the fluid is now modelled as a Poisseuille flow with an open boundary.

2.2 Poisseuille flow

Incompressible Navier-Stokes equation:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\varrho} + \eta \Delta \mathbf{v} + \mathbf{g} \tag{1}$$

Taking the z-axis to be along the symmetry axis of the pipe we have

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ v(x,y) \end{pmatrix} \tag{2}$$

and thus

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = v(x,y)\partial_z v(x,y) \equiv 0 \tag{3}$$

Assuming the flow to be stationary follows from the above used assumption that the velocity does not depend on z. This assumption is not selfevident as the fluid is freely falling. However it seems reasonable to assume that after a short (?) acceleration time the velocity stays constant equating gravity with friction forces.

Experimental evidence for or against this assumption can be gathered as follows. A change in the velocity of an incompressible fluid is possible if the cross section of the flow shrinks or grows. This is however hard to observe in our case as the width of the flow is rather small.

We thus get two equations:

$$0 = \partial_x P = \partial_y P \tag{4}$$

$$\frac{\partial_z P}{\varrho} = \Delta_{xy} v(x,y) - g \tag{5}$$

As the first equation states that P is independent of x and y the two sides of the second equation must be constant. We choose this constant to be $\frac{\delta P}{\ell}$:

$$\Delta_{xy} v(x,y) = \frac{1}{r}\partial_r r \partial_r v(r) = \frac{\delta P}{\ell \varrho} + g =: c \tag{6}$$

$$r\partial_r v(r) = c\frac{r^2}{2} + d \tag{7}$$

$$v(r) = c\frac{r^2}{4} + D \ln r + E \tag{8}$$

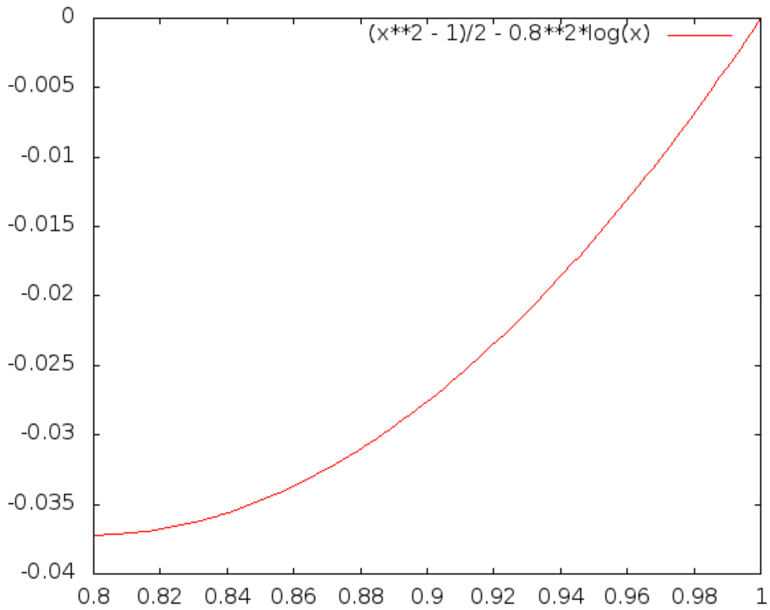

Figure 1: Plot of $v(\beta)/v_0$ for $\alpha = 0.8$.

Call the inner radius of the pipe R and the radius of the rising air "bubble" R_A. We choose $v(R) \equiv 0$ and $v'(R_\mathrm{A}) = 0$ on the free surface:

$$E = -c\frac{R^2}{4} - D \ln R \tag{9}$$

$$v'(R_\mathrm{A}) = 0 = c\frac{R_\mathrm{A}}{2} + D\frac{1}{R_\mathrm{A}} \tag{10}$$

$$-c\frac{R_\mathrm{A}^2}{2} = D \tag{11}$$

$$\Rightarrow v(r) = \frac{c}{2}\left(\frac{r^2 - R^2}{2} - R_\mathrm{A}^2 \ln \frac{r}{R}\right) \tag{12}$$

$$\tag{13}$$

After defining dimensionless parameters $\alpha := \frac{R_\mathrm{A}}{R}$, $\beta := \frac{r}{R}$ and $v_0 := \frac{R^2 c}{2}$, we get

$$v(\beta) = v_0 \left(\frac{\beta^2 - 1}{2} - \alpha^2 \ln \beta\right) \tag{14}$$

$$\tag{15}$$

We can now calculate the total outflow of volume per unit time:

$$\pi\bar{v}(R^2 - R_\mathrm{A}^2) := \int_{R_\mathrm{A}}^{R} \mathrm{d}r\, 2\pi r v(r) = 2\pi v_0 R^2 \int_\alpha^1 \mathrm{d}\beta\, \beta \left(\frac{\beta^2 - 1}{2} - \alpha^2 \ln \beta\right) \tag{16}$$

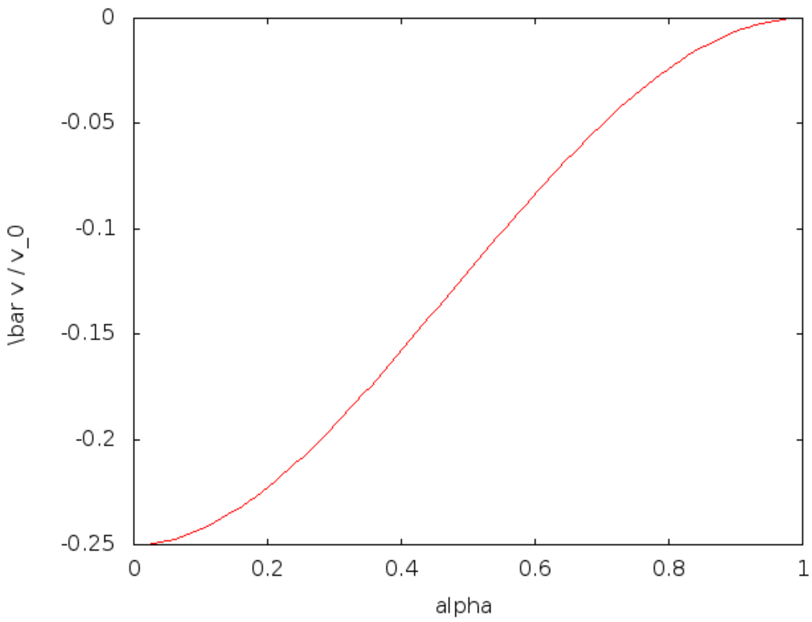

Figure 2: Plot of $\bar{v}/v_0$.

Using

$$\int x \ln x \, \mathrm{d}x = \frac{x^2}{2} \ln x - \frac{x^2}{4} \tag{17}$$

we get

$$\bar{v}(R^2 - R_\mathrm{A}^2) = 2v_0 R^2 \left(-\frac{1}{8}(\alpha^2 - 1)^2 - \alpha^2 \left[-\frac{1}{4} - \frac{\alpha^2}{2} \ln \alpha + \frac{\alpha^2}{4} \right] \right) \tag{18}$$

$$\bar{v} = v_0 \frac{1}{1 - \alpha^2} \left(\frac{-3\alpha^4 + 4\alpha^2 - 1}{4} + \alpha^4 \ln \alpha \right) \tag{19}$$

$$\bar{v} = v_0 \frac{1}{1 - \alpha^2} \left(-\frac{(3\alpha^2 - 1)(\alpha^2 - 1)}{4} + \alpha^4 \ln \alpha \right) \tag{20}$$

2.3 Continuity

2.3.1 Velocity of the bubble

By continuity the fluid volume $\pi(R^2 - R_\mathrm{A}^2)|\bar{v}|$ leaving the pipe per unit time must be replaced by the same amount of air volume, thus yielding

$$\pi(R^2 - R_\mathrm{A}^2)|\bar{v}| = \pi R_\mathrm{A}^2 v_\mathrm{A} \tag{21}$$

or

$$(1-\alpha^2)|\bar{v}| = \alpha^2 v_{\mathrm{A}} \tag{22}$$

Given α we can thus calculate v_{A}.

2.3.2 Change of cross section

Assuming that structurally the same flow profile can be found in two parts of the same pipe, which now however shall have different widths in those parts, we can get another continuity expression. This assumption is wrong for strong changes of profile, where we emprirically state the appearence of cluck-cluck.

Call (thinking of a wine bottle which consists of a narrow pipe, a wide pipe and a smooth connection inbetween, which we will ignore) the two inner radii of the pipes $R^{\pm}$, the two bubble radii $R_{\mathrm{A}}^{\pm}$, the two velocity profiles $v^{\pm}(r)$, the mean velocities $\bar{v}^{\pm}$ and the two velocities of the bubble $v_{\mathrm{A}}^{\pm}$ respectively. Then we know:

$$R_{\mathrm{A}}^{+2} v_{\mathrm{A}}^{+} = R_{\mathrm{A}}^{-2} v_{\mathrm{A}}^{-} \tag{23}$$

$$R_{\mathrm{A}}^{+2} \frac{R^{+2} - R_{\mathrm{A}}^{+2}}{R_{\mathrm{A}}^{+2}} \bar{v}^{+} = R_{\mathrm{A}}^{-2} \frac{R^{-2} - R_{\mathrm{A}}^{-2}}{R_{\mathrm{A}}^{-2}} \bar{v}^{-} \tag{24}$$

$$(R^{+2} - R_{\mathrm{A}}^{+2})\bar{v}^{+} = (R^{-2} - R_{\mathrm{A}}^{-2})\bar{v}^{-} \tag{25}$$

$$\tag{26}$$

Given R_{A}^{+} we can solve this equation for R_{A}^{-} and thus have a relationship between v_{A}^{+} and v_{A}^{-}.

$$(R^{+2} - R_{\mathrm{A}}^{+2}) \frac{R^{+2} c}{2} \frac{1}{1-\alpha^{+2}} \left(-\frac{(3\alpha^{+2}-1)(\alpha^{+2}-1)}{4} + \alpha^{+4} \ln \alpha^{+} \right) = \tag{27}$$

$$(R^{-2} - R_{\mathrm{A}}^{-2}) \frac{R^{-2} c}{2} \frac{1}{1-\alpha^{-2}} \left(-\frac{(3\alpha^{-2}-1)(\alpha^{-2}-1)}{4} + \alpha^{-4} \ln \alpha^{-} \right) \tag{28}$$

$$(1-\alpha^{+2}) R^{+4} \frac{1}{1-\alpha^{+2}} \left(-\frac{(3\alpha^{+2}-1)(\alpha^{+2}-1)}{4} + \alpha^{+4} \ln \alpha^{+} \right) = \tag{29}$$

$$(1-\alpha^{-2}) R^{-4} \frac{1}{1-\alpha^{-2}} \left(-\frac{(3\alpha^{-2}-1)(\alpha^{-2}-1)}{4} + \alpha^{-4} \ln \alpha^{-} \right) \tag{30}$$

$$\tag{31}$$

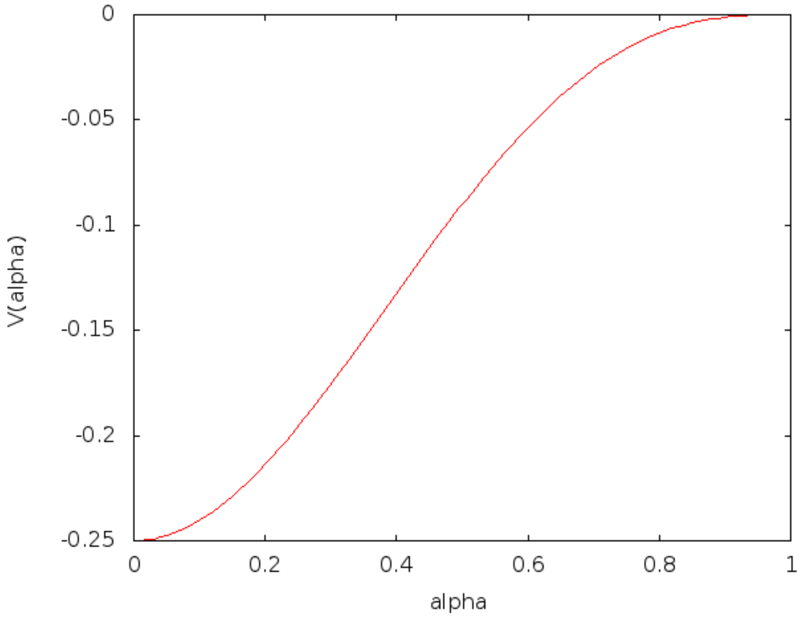

Figure 3: Plot of $V(\alpha)$.

We have eliminated c, assuming $c^+ = c^-$. Defining

$$V(\alpha) := -\frac{(3\alpha^2 - 1)(\alpha^2 - 1)}{4} + \alpha^4 \ln \alpha \tag{32}$$

$$R^{+4} V(\alpha^+) = R^{-4} V(\alpha^-) \tag{33}$$

$$\tag{34}$$

We observe that $V(\alpha)$ is strictly monotonous. Thus if we have

$$R^+ > R^- \tag{35}$$

$$\frac{R^{+4}}{R^{-4}} = \frac{V(\alpha^-)}{V(\alpha^+)} > 1 \tag{36}$$

$$V(\alpha^-) > V(\alpha^+) \tag{37}$$

$$\alpha^- > \alpha^+ \tag{38}$$

If the cross section becomes wider, the width α becomes smaller, putting through the same volume per unit of time.

2.4 Surface tension

Surface tension dominates the behaviour of narrow pipes. Energy minimization should in general lead to an outflow of the fluid. However this is connected with a creation of surface in order to let air in. The energy necessary for a spherical air bubble of radius r

is

$$V_\sigma = \sigma \cdot 4\pi r^2 \tag{39}$$

σ has for water-air interfaces the numerical value of $\sim 70\,\frac{\mathrm{mJ}}{\mathrm{m}^2}$. The energy won by letting in this air bubble is the potential energy of the same volume of water:

$$V_g = \frac{4}{3}\pi r^3 \varrho g h \tag{40}$$

Assuming $h \simeq r$ water will only flow out if

$$V_\sigma < V_g \tag{41}$$

$$\frac{3\sigma}{\varrho g} < r^2 \tag{42}$$

$$\tag{43}$$

The critical value is of the order of

$$\sqrt{\frac{3 \cdot 70 \cdot 10^{-3}}{1000 \cdot 10}} \simeq \sqrt{20 \cdot 10^{-6}} \simeq 4 \cdot 10^{-3}\,\mathrm{m} \tag{44}$$

Thus in pipes of radius less then $\sim 4\,\mathrm{mm}$ water will not flow out. This is confirmed by experiments.

3 Instabilities in non-constant cross sections