Some derivatives and their transformation properties

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Every derivative of a vector(field) V^μ with respect to another is necessarily a two-index-structure. However many of those misleadingly do not transform as tensors.

Contents

1 Why is a simple coordinate derivative not a tensor?

Consider

$$V^\mu{}_{,\nu} = \partial_\nu V^\mu = \frac{\partial}{\partial x^\nu} V^\mu$$

This is not a tensor:

$$\frac{\partial}{\partial x^{\bar\nu}} V^{\bar\mu} = \frac{\partial x^\sigma}{\partial x^{\bar\nu}} \frac{\partial}{\partial x^\sigma} V^{\bar\mu} = \frac{\partial x^\sigma}{\partial x^{\bar\nu}} \frac{\partial}{\partial x^\sigma} \left(\frac{\partial x^{\bar\mu}}{\partial x^\rho} V^\rho \right) = \underbrace{\frac{\partial x^\sigma}{\partial x^{\bar\nu}} \frac{\partial x^{\bar\mu}}{\partial x^\rho} \frac{\partial}{\partial x^\sigma} V^\rho}_{\text{OK!}} + \underbrace{\frac{\partial x^\sigma}{\partial x^{\bar\nu}} \frac{\partial}{\partial x^\sigma} \frac{\partial x^{\bar\mu}}{\partial x^\rho} V^\rho}_{\text{extra term!}} \tag{1}$$

2 Why are Christoffel symbols not tensors?

Christoffel symbols are defined as

$$\nabla_{\partial_\mu} \partial_\nu = \Gamma^\sigma_{\mu\nu} \partial_\sigma$$

They transform as

$$\Gamma^{\bar\sigma}_{\bar\mu\bar\nu}\partial_{\bar\sigma} = \Gamma^{\bar\sigma}_{\bar\mu\bar\nu}\frac{\partial x^\alpha}{\partial x^{\bar\sigma}}\frac{\partial}{\partial x^\alpha} = \nabla_{\frac{\partial x^\rho}{\partial x^{\bar\mu}}\frac{\partial}{\partial x^\rho}}\left(\frac{\partial x^\pi}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\pi}\right) = \frac{\partial x^\rho}{\partial x^{\bar\mu}}\left(\frac{\partial}{\partial x^\rho}\frac{\partial x^\pi}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\pi} + \frac{\partial x^\pi}{\partial x^{\bar\nu}}\nabla_{\frac{\partial}{\partial x^\rho}}\frac{\partial}{\partial x^\pi}\right) \tag{2}$$

$$= \frac{\partial x^\rho}{\partial x^{\bar\mu}}\frac{\partial}{\partial x^\rho}\frac{\partial x^\alpha}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\alpha} + \frac{\partial x^\rho}{\partial x^{\bar\mu}}\frac{\partial x^\pi}{\partial x^{\bar\nu}}\Gamma^\alpha_{\rho\pi}\frac{\partial}{\partial x^\alpha} \tag{3}$$

$$\Gamma^{\bar\sigma}_{\bar\mu\bar\nu} = \underbrace{\frac{\partial x^{\bar\sigma}}{\partial x^\alpha}\frac{\partial x^\rho}{\partial x^{\bar\mu}}\frac{\partial}{\partial x^\rho}\frac{\partial x^\alpha}{\partial x^{\bar\nu}}}_{\text{extra term!}} + \underbrace{\frac{\partial x^{\bar\sigma}}{\partial x^\alpha}\frac{\partial x^\rho}{\partial x^{\bar\mu}}\frac{\partial x^\pi}{\partial x^{\bar\nu}}\Gamma^\alpha_{\rho\pi}}_{\text{fine!}} \tag{4}$$

3 Why is the covariant derivative a tensor?

The covariant derivative corrects for both effects:

$$\nabla_{\frac{\partial}{\partial x^\nu}}V = (V^\mu{}_{,\nu} + V^\alpha\Gamma^\mu_{\nu\alpha})\frac{\partial}{\partial x^\mu} = V^\mu{}_{;\nu}\partial_\mu \tag{5}$$

$$V^{\bar\mu}{}_{;\bar\nu} = V^{\bar\mu}{}_{,\bar\nu} + V^{\bar\alpha}\Gamma^{\bar\mu}_{\bar\nu\bar\alpha} \tag{6}$$

$$= \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial x^{\bar\mu}}{\partial x^\rho}\frac{\partial}{\partial x^\sigma}V^\rho + \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\sigma}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho + \frac{\partial x^{\bar\alpha}}{\partial x^\beta}V^\beta\left(\frac{\partial x^{\bar\mu}}{\partial x^\delta}\frac{\partial x^\rho}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\rho}\frac{\partial x^\delta}{\partial x^{\bar\alpha}} + \frac{\partial x^{\bar\mu}}{\partial x^\delta}\frac{\partial x^\rho}{\partial x^{\bar\nu}}\frac{\partial x^\pi}{\partial x^{\bar\alpha}}\Gamma^\delta_{\rho\pi}\right) \tag{7}$$

$$= \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho{}_{,\sigma} + V^\beta\delta^\pi_\beta\frac{\partial x^{\bar\mu}}{\partial x^\delta}\frac{\partial x^\rho}{\partial x^{\bar\nu}}\Gamma^\delta_{\rho\pi} + \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\sigma}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho + V^\beta\delta^\delta_\beta\frac{\partial x^{\bar\mu}}{\partial x^\delta}\frac{\partial x^\rho}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\rho} \tag{8}$$

$$= \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial x^{\bar\mu}}{\partial x^\rho}(V^\rho{}_{,\sigma} + V^\pi\Gamma^\rho_{\sigma\pi}) + \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\sigma}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho + V^\beta\frac{\partial x^{\bar\mu}}{\partial x^\delta}\frac{\partial x^\rho}{\partial x^{\bar\nu}}\frac{\partial x^{\bar\alpha}}{\partial x^\beta}\frac{\partial}{\partial x^\rho}\frac{\partial x^\delta}{\partial x^{\bar\alpha}} \tag{9}$$

$$= \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho{}_{;\sigma} + \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\sigma}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho + V^\beta\frac{\partial x^{\bar\mu}}{\partial x^\delta}\frac{\partial x^\rho}{\partial x^{\bar\nu}}\left(\frac{\partial}{\partial x^\rho}\left\{\frac{\partial x^{\bar\alpha}}{\partial x^\beta}\frac{\partial x^\delta}{\partial x^{\bar\alpha}}\right\} - \frac{\partial}{\partial x^\rho}\frac{\partial x^{\bar\alpha}}{\partial x^\beta}\frac{\partial x^\delta}{\partial x^{\bar\alpha}}\right) \tag{10}$$

$$= \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho{}_{;\sigma} + \frac{\partial x^\rho}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\rho}\frac{\partial x^{\bar\mu}}{\partial x^\beta}V^\beta - V^\beta\delta^{\bar\mu}_{\bar\alpha}\frac{\partial x^\rho}{\partial x^{\bar\nu}}\frac{\partial}{\partial x^\rho}\frac{\partial x^{\bar\alpha}}{\partial x^\beta} \tag{11}$$

$$= \frac{\partial x^\sigma}{\partial x^{\bar\nu}}\frac{\partial x^{\bar\mu}}{\partial x^\rho}V^\rho{}_{;\sigma} \tag{12}$$

4 Why is the Lie derivative not a tensor?

Using a metric connection you can write

$$\mathcal{L}_{\frac{\partial}{\partial x^\nu}}V = \nabla_{\frac{\partial}{\partial x^\nu}}V - \nabla_V\frac{\partial}{\partial x^\nu} = V^\mu{}_{;\nu}\partial_\mu - V^\mu\nabla_{\frac{\partial}{\partial x^\mu}}\frac{\partial}{\partial x^\nu} = \left(V^\mu{}_{;\nu} - V^\beta\Gamma^\mu_{\beta\nu}\right)\frac{\partial}{\partial x^\mu} \tag{13}$$

However the second summand inside the brackets is a sum of non-tensorial quantities, and thus cannot be a tensor for arbitrary V.