Curvature of a plane curve

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1 Two derivations of the curvature formula

Given a twice differentiable function $f(x)$ and we would like to know its curvature radius at x.

1.1 Circle fit

The curvature radius is interpreted as the radius of a circle fit, i.e. of a circle that shares the function value and the first and second derivatives in $(x, f(x))$ with the function. This circle is described by $(y - M_y)^2 + (x - M_x)^2 = R^2$ or $y = \pm\sqrt{R^2 - (x - M_x)^2} + M_y$. Derivatives:

$$\frac{dy}{dx} = \pm\frac{M_x - x}{\sqrt{R^2 - (x - M_x)^2}}$$
$$\frac{d^2y}{dx^2} = \mp\frac{R^2}{(R^2 - (x - M_x)^2)^{3/2}}$$

Thus

$$f(x) = \pm\sqrt{R^2 - (x - M_x)^2} + M_y$$
$$f'(x) = \pm\frac{M_x - x}{\sqrt{R^2 - (x - M_x)^2}}$$
$$f''(x) = \mp\frac{R^2}{(R^2 - (x - M_x)^2)^{3/2}}$$

The two last equations depend only on the seeked R and on M_x, that has to be eliminated:

$$f'^2(x) = \frac{(M_x - x)^2}{R^2 - (x - M_x)^2}$$
$$f''^2(x) = \frac{R^4}{(R^2 - (x - M_x)^2)^3}$$

With the second equation:

$$f''^{2/3}(x) = \frac{R^{4/3}}{R^2 - (x - M_x)^2}$$
$$R^2 - (x - M_x)^2 = \frac{R^{4/3}}{f''^{2/3}(x)}$$
$$(x - M_x)^2 = R^2 - \frac{R^{4/3}}{f''^{2/3}(x)}$$

Insert into the first equation:

$$\begin{aligned} f'^2(x) &= \frac{R^2 - \frac{R^{4/3}}{f''^{2/3}(x)}}{R^2 - R^2 + \frac{R^{4/3}}{f''^{2/3}(x)}} \\ &= \frac{R^2 f''^{2/3}(x) - R^{4/3}}{R^{4/3}} \\ &= R^{2/3} f''^{2/3}(x) - 1 \\ \frac{f'^2(x) + 1}{f''^{2/3}(x)} &= R^{2/3} \end{aligned}$$

Thus one finds $\boxed{R = \frac{1}{\kappa} = \frac{(f'^2(x) + 1)^{3/2}}{f''(x)}}$.

1.2 Infinitesimal circle element

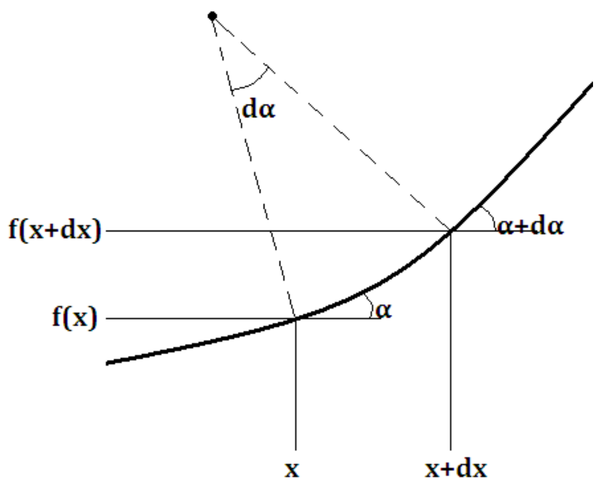

Figure 1: Illustration of the geometry.

The idea is to identify the curvature radius with the one that satisfies the infinitesimal equation $dl = R d\alpha$. As is well known

$$\tan(\alpha + d\alpha) = \frac{\tan\alpha + d\alpha}{1 - d\alpha \tan\alpha}$$

Here we have:

$$\tan\alpha = f'(x)$$
$$\tan(\alpha + d\alpha) = \frac{\tan\alpha + d\alpha}{1 - d\alpha\tan\alpha} = \frac{f'(x) + d\alpha}{1 - d\alpha f'(x)} = f'(x + dx)$$

The last equation gives

$$f'(x) + d\alpha = f'(x + dx) - f'^2(x)d\alpha$$
$$d\alpha = \frac{f'(x + dx) - f'(x)}{1 + f'^2(x)} = dx\frac{f''(x)}{1 + f'^2(x)}$$

With $dl = \sqrt{1 + f'^2(x)}dx$ one finds

$$R = \frac{dl}{d\alpha} = \frac{\sqrt{1 + f'^2(x)}(1 + f'^2(x))}{f''(x)}$$

And thus again $\boxed{R = \frac{1}{\kappa} = \frac{(f'^2(x) + 1)^{3/2}}{f''(x)}}$

2 Curvature as a rotation-invariant functional

When subject to rotation of its graph in general both the function values and the values of the derivatives are expected to change. Consider in the following a curve

$$\gamma : \mathbb{R} \to \mathbb{R}^2 \tag{1}$$
$$t \mapsto \gamma(t) = (x(t), y(t))^{\mathrm{T}} \tag{2}$$

with again sufficiently many existing derivatives.

The rotated curve γ^* is given by

$$\gamma^* = M\gamma = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \simeq \begin{pmatrix} 1 & -\phi \\ \phi & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - \phi y \\ \phi x + y \end{pmatrix} \tag{3}$$

It is sufficient to restrict oneself to infinitesimal rotations as we are interested in invariants.

Assuming γ to be (locally) the graph of a function $y = f(x)$ one can now derive the transformation laws of f and its derivatives under infintesimal rotations. Surely one sees already that $f \mapsto f + \phi x$. Moreover

$$f' = \frac{\partial_t y}{\partial_t x} \mapsto \frac{\phi\partial_t x + \partial_t y}{\partial_t x - \phi\partial_t y} \simeq \frac{(\phi\partial_t x + \partial_t y)(\partial_t x + \phi\partial_t y)}{(\partial_t x)^2} \simeq \frac{\partial_t y}{\partial_t x} + \frac{(\partial_t x)^2 + (\partial_t y)^2}{(\partial_t x)^2}\phi \tag{4}$$
$$= f' + \underbrace{(1 + f'^2)}_{=:\sigma}\phi \tag{5}$$

$$f'' = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial_t x\partial_t^2 y - \partial_t y\partial_t^2 x}{(\partial_t x)^3} \tag{6}$$
$$\mapsto \frac{(\partial_t x - \phi\partial_t y)(\phi\partial_t^2 x + \partial_t^2 y) - (\phi\partial_t x + \partial_t y)(\partial_t^2 x - \phi\partial_t^2 y)}{(\partial_t x - \phi\partial_t y)^3} \tag{7}$$
$$\simeq \frac{\partial_t x\partial_t^2 y - \partial_t y\partial_t^2 x}{(\partial_t x)^4}(\partial_t x + 3\phi\partial_t y) \tag{8}$$
$$= \frac{\partial_t x\partial_t^2 y - \partial_t y\partial_t^2 x}{(\partial_t x)^3} + 3\frac{\partial_t x\partial_t^2 y - \partial_t y\partial_t^2 x}{(\partial_t x)^3}\frac{\partial_t y}{\partial_t x}\phi = f'' + 3f''f'\phi \tag{9}$$

Consider now a functional $S[f, f', f'']$ and its value under an infinitesimal rotation

$$S[f + \phi x, f' + \sigma\phi, f'' + 3f''f'\phi] \simeq S[f, f', f''] + \phi x \partial_f S + \sigma\phi\partial_{f'} S + 3f''f'\phi\partial_{f''} S \qquad (10)$$

Invariance is obtained with

$$x\partial_f S + \sigma\partial_{f'} S + 3f''f'\partial_{f''} S = 0 \qquad (11)$$

2.1 Special cases

2.1.1 Curvature

Consider S to be linear in f'' and indendependent of f:

$$S = f'' \cdot P(f') \qquad (12)$$

Then

$$(1 + f'^2)f''P' + 3f''f'P = 0 \qquad (13)$$

$$\frac{P'}{P} = -\frac{3f'}{1 + f'^2} \qquad (14)$$

$$\ln P + \text{const} = -\frac{3}{2}\ln(1 + f'^2) \qquad (15)$$

$$P = \frac{\text{const}}{(1 + f'^2)^{3/2}} \qquad (16)$$

which gives the well-known formula for the curvature back.